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II. *On the Stresses in the Neighbourhood of a Circular Hole in a Strip Under Tension.*

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(*Communicated by* L. N. G. Filon, *F.R.S.*)

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Introduction.

The problem of determining the stresses in a plate under tension when the material is pierced by one or more circular holes is one of both theoretical interest and practical importance. Provided that the plate may be regarded as infinitely extended in two dimensions, the solution for a single hole is easily found and is well known.* The presence of the hole leads to the occurrence of stresses equal to three times the tension at infinity, these maximum stresses occurring at the edge of the hole and on the diameter perpendicular to the direction of the applied tension.

More general stress systems, corresponding to the presence of tractions at the edge of the hole, may be studied by similar methods, not only when the plate is infinite but also when there is a second circular boundary concentric with the first.† A number of special solutions for the infinite plate have recently been published by Bickley.‡

The solution for a semi-infinite plate with one circular hole was obtained by Jeffery, using bipolar co-ordinates,§ which may be applied also to the case of an infinite plate pierced by two holes.

In the present paper the plate is supposed bounded by two parallel edges, and to contain a hole mid-way between the edges, and a solution of the tension problem is sought by successive approximation. The analysis is, however, first developed for the more general case of any stress system symmetrical both about the axis of the strip and about the diameter of the hole perpendicular to the axis. General formulæ are obtained giving the coefficients of each approximation linearly in terms of those of the preceding approximation. The process used is analogous to the alternating process of Schwarz,||

* See, for example, Prescott, "Applied Elasticity," p. 361 (London, 1924). An elementary solution, ascribed to Southwell, is given by Morley, "Strength of Materials," 4th edn. (London, 1916) and later editions, Appendix.

† Filon, "Inst. Civil Eng., Selected Papers," No. 12 (1924).

‡ 'Phil. Trans.,' A, vol. 227, pp. 383–415 (1928).

§ 'Phil. Trans.,' A, vol. 221, pp. 265–293 (1920).

|| Picard, "Traité d'Analyse," vol. 2, chap. 3, gives an account of the method with references to the original memoirs.

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which has been applied by PICARD to the general partial differential equation of the second order.* No general theory for the bi-harmonic equation appears to have been published. The problem of convergence is a difficult one, though it may perhaps be solved by an extension of the method used here to establish the convergence of some subsidiary series. It would be of great theoretical interest, but the lack of a general theory is not serious in practice, as the degree of approximation can be tested at every stage by examining the residual tractions on the edge of the strip. From this practical standpoint the convergence may be said to be rapid when the diameter of the hole is not more than half that of the strip.

Apart from the heavy nature of the algebra involved, there should be no essential difficulty in extending the analysis to cover unsymmetrical stress-systems, and a wide variety of problems will then be brought within its reach.

Statement of Method.

We consider a strip of isotropic, elastic material, bounded in the x, y-plane by the lines $y = \pm b$, and of uniform thickness perpendicular to this plane. The hole will

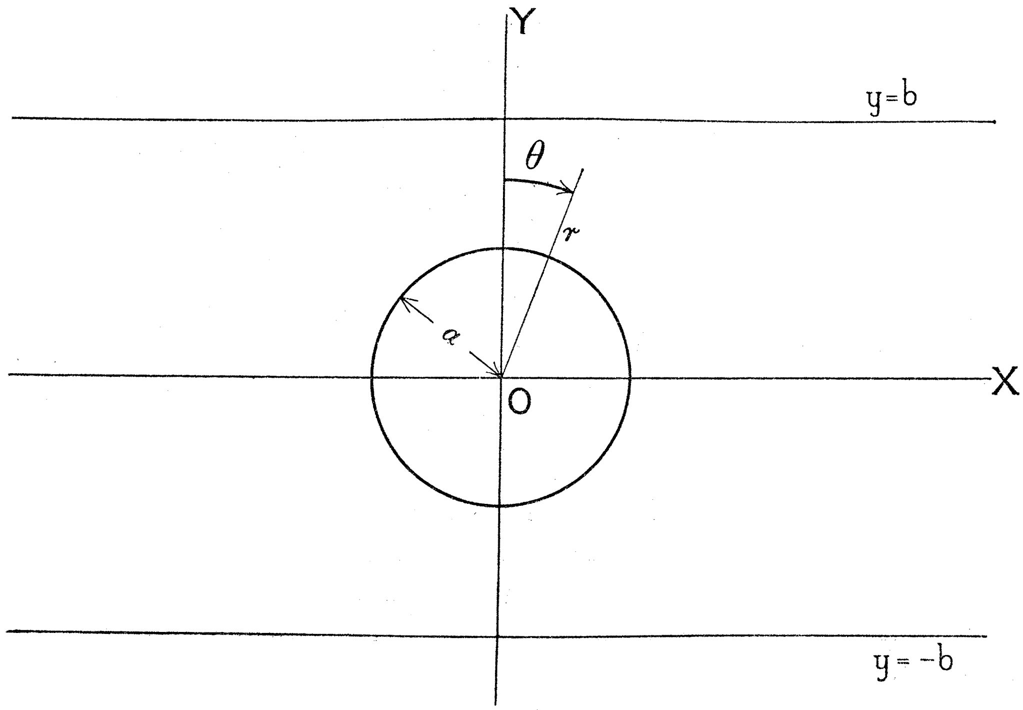

FIG. 1.

be supposed to have its centre at the origin, and to be of radius $a\,(< b)$. Polar co-ordinates (r, θ) will also be used, and it will be convenient to take the initial line along the y-axis and the positive direction of θ clockwise (fig. 1).

* ' J. Mathématique,' vol. 5, pp. 145–210 (1890).

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Then the relation between the two systems of co-ordinates is

$$\left.\begin{aligned} x &= r \sin \theta \\ y &= r \cos \theta \end{aligned}\right\} . \qquad (1)$$

We shall also write

$$\lambda = a/b, \quad \xi = x/b, \quad \eta = y/b, \quad \rho = r/b, \qquad (2)$$

so that ξ, η, ρ are co-ordinates measured in a unit equal to half the width of the strip, and λ is the radius of the hole measured in the same unit. In practice λ would never be greater than $\frac{1}{2}$, and greater values than this will not be considered in the present paper.

The material will be supposed to be in a state of generalised plane stress,* specified by a stress function χ. This function must satisfy the following conditions :—

(*a*) At all points within the material

$$\nabla^4 \chi = 0 ; \qquad (3)$$

(*b*) the stresses $\widehat{xx} = \dfrac{1}{b^2}\dfrac{\partial^2\chi}{\partial\eta^2}$, $\widehat{yy} = \dfrac{1}{b^2}\dfrac{\partial^2\chi}{\partial\xi^2}$, $\widehat{xy} = -\dfrac{1}{b^2}\dfrac{\partial^2\chi,}{\partial\xi\,\partial\eta}$, tend to definite values at infinity; in particular, in the tension problem, $\widehat{xx} \to T$ (constant), $\widehat{yy} \to 0$, $\widehat{xy} \to 0$, when $x \to \pm\infty$.

(*c*) On the straight edges, $\eta = \pm 1$,

$$\frac{\partial^2\chi}{\partial\xi^2} = \frac{\partial^2\chi}{\partial\xi\,\partial\eta} = 0 ;$$

(*d*) At the rim of the hole, $\rho = \lambda$.

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{b^2}\left[\frac{1}{\rho^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial\chi}{\partial\rho}\right] = f(\theta) \\ \widehat{r\theta} &= -\frac{1}{b^2}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\chi}{\partial\theta}\right) = \phi(\theta) \end{aligned}\right\} , \qquad (4)$$

where $f(\theta)$ and $\phi(\theta)$ are given functions, both even and both symmetrical about $\theta = \frac{1}{2}\pi$; in particular, in the tension problem, the rim of the hole is free from stress and $f(\theta)$ and $\phi(\theta)$ both vanish.

To satisfy these conditions, we write

$$\chi = \chi_0' + \chi_0 + \chi_1 + \chi_2 + \ldots , \qquad (5)$$

* LOVE, "Theory of Elasticity," 4th edn., p. 138 (Cambridge, 1927). The theory was given by FILON, 'Phil. Trans.,' A, vol. 201, p. 67 (1903).

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where the terms of the series are each, separately, solutions of the biharmonic equation (3) and have, in addition, the following properties.

χ_0' gives the stresses at infinity and none on the edges $\eta = \pm 1$.

$\chi_0' + \chi_0$ satisfies the conditions at the rim of the hole and at infinity, but not on the edges, *i.e.*, it is the solution for an infinite plate. This will be supposed known, χ_0 being expressible in the form

$$\chi_0 = - D_0^{(0)} \log \rho + \sum_{n=1}^{\infty} \left(\frac{D_{2n}^{(0)}}{\rho^{2n}} + \frac{E_{2n}^{(0)}}{\rho^{2n-2}} \right) \cos 2n\theta. \quad \ldots \ldots \quad (6)$$

χ_1 cancels the stresses due to χ_0 on the edges $\eta = \pm 1$, but introduces stresses on the rim of the hole ; χ_2 cancels these, but again produces stresses on the edges.

More generally, $\chi_{2r} + \chi_{2r+1}$ gives zero stresses on $\eta = \pm 1$, while $\chi_{2r-1} + \chi_{2r}$ gives zero stresses over $\rho = \lambda$. For definiteness we add the conditions that the stresses due to χ_{2r} shall tend to 0 when $\rho \to \infty$, while those due to χ_{2r+1} shall be finite at the origin and throughout the finite part of the strip. All the stress functions are then fully determinate, and χ will satisfy the required conditions, provided that the series in (5) and its derivatives up to the fourth are uniformly convergent.

If the series is truncated after χ_{2r} it will give a value of χ satisfying all the conditions except those on the edges $\eta = \pm 1$. If the residual tractions due to χ_{2r} are small enough, this value of χ is, for practical purposes, the value required. Similarly, if the series is truncated after χ_{2r+1} the resulting value of χ satisfies all the conditions except those at the rim of the hole $\rho = \lambda$. If the additional tractions due to χ_{2r+1} are small enough, the solution is again sufficient in practice. In neither case is there any problem of convergence ; but it is clear that, if the method is to be practicable, it is necessary that there should be terms fairly early in the series which correspond to tractions of negligible amount. It will appear later that, if λ does not exceed $0 \cdot 5$, it is never necessary to proceed beyond χ_8, while if λ is less than $0 \cdot 25$ it is possible to stop at χ_2. Values of λ much larger than $0 \cdot 5$ would lead to very laborious computations.

Fundamental Formulæ.

We suppose that the cycle of operations described in the preceding section has been carried out r times, and that we have arrived at the value of the stress-function χ_{2r}. It is the object of the present section to derive equations from which χ_{2r+1} and χ_{2r+2} may be calculated. This completes a fresh cycle, and the process may then be repeated indefinitely. The value of χ_{2r} will be assumed to be given in the form

$$\chi_{2r} = - D_0^{(r)} \log \rho + \sum_{n=1}^{\infty} \left(\frac{D_{2n}^{(r)}}{\rho^{2n}} + \frac{E_{2n}^{(r)}}{\rho^{2n-2}} \right) \cos 2n\theta, \quad \ldots \ldots \quad (7)$$

and the aim is to find the coefficients in a similar series for χ_{2r+2}.

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For simplicity of writing, it will be convenient to omit the suffix (r) in all the coefficients until it becomes necessary to distinguish them from those of the χ_{2r+2} series. Making this temporary simplification of notation, we have for the stresses due to χ_{2r},

$$\widehat{rr} = \frac{1}{b^2}\left(\frac{1}{\rho}\frac{\partial \chi_{2r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \chi_{2r}}{\partial \theta^2}\right)$$

$$= \frac{1}{b^2}\left[-\frac{D_0}{\rho^2} - 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}} + \frac{(n+1)(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos 2n\theta\right] \quad . \quad . \quad . \quad . \quad (8)$$

$$\widehat{r\theta} = -\frac{1}{b^2}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \chi_{2r}}{\partial \theta}\right) = -\frac{2}{b^2}\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}} + \frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\sin 2n\theta, \quad . \quad . \quad (9)$$

$$\widehat{\theta\theta} = \frac{1}{b^2}\frac{\partial^2 \chi_{2r}}{\partial \rho^2} = \frac{1}{b^2}\left[\frac{D_0}{\rho^2} + 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}} + \frac{(n-1)(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos 2n\theta\right] \quad . \quad . \quad (10)$$

The stresses relative to the Cartesian axes are now given by the equations

$$\left.\begin{aligned} \widehat{xx} &= \widehat{rr}\sin^2\theta + 2\widehat{r\theta}\sin\theta\cos\theta + \widehat{\theta\theta}\cos^2\theta \\ \widehat{xy} &= (\widehat{rr} - \widehat{\theta\theta})\sin\theta\cos\theta + \widehat{r\theta}(\cos^2\theta - \sin^2\theta) \\ \widehat{yy} &= \widehat{rr}\cos^2\theta - 2\widehat{r\theta}\sin\theta\cos\theta + \widehat{\theta\theta}\sin^2\theta \end{aligned}\right\},$$

which differ slightly from the standard forms on account of the way in which θ has been measured. These we re-write in the forms

$$\left.\begin{aligned} \widehat{xx} &= \tfrac{1}{2}\{(\widehat{rr} + \widehat{\theta\theta}) - (\widehat{rr} - \widehat{\theta\theta})\cos 2\theta + 2\widehat{r\theta}\sin 2\theta\} \\ \widehat{yy} &= \tfrac{1}{2}\{(\widehat{rr} + \widehat{\theta\theta}) + (\widehat{rr} - \widehat{\theta\theta})\cos 2\theta - 2\widehat{r\theta}\sin 2\theta\} \\ \widehat{xy} &= \tfrac{1}{2}\{(\widehat{rr} - \widehat{\theta\theta})\sin 2\theta + 2\widehat{r\theta}\cos 2\theta\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now it is easily seen that

$$\tfrac{1}{2}(\widehat{rr} - \widehat{\theta\theta}) = -\frac{1}{b^2}\left[\frac{D_0}{\rho^2} + 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}} + \frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos 2n\theta\right].$$

Hence

$$\tfrac{1}{2}(\widehat{rr} - \widehat{\theta\theta})\cos 2\theta - \widehat{r\theta}\sin 2\theta$$
$$= -\frac{1}{b^2}\left[\frac{D_0}{\rho^2}\cos 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}} + \frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\theta\right]$$

and

$$\tfrac{1}{2}(\widehat{rr} - \widehat{\theta\theta})\sin 2\theta + \widehat{r\theta}\cos 2\theta$$
$$= -\frac{1}{b^2}\left[\frac{D_0}{\rho^2}\sin 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}} + \frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\sin(2n+2)\theta\right].$$

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Also

$$\tfrac{1}{2}(\widehat{rr}+\widehat{\theta\theta}) = -\frac{2}{b^2}\sum_{n=1}^{\infty}\frac{(2n-1)\,\mathrm{E}_{2n}}{\rho^{2n}}\cos 2n\theta.$$

Substituting these values into (11), we obtain

$$\widehat{xx} = \frac{1}{b^2}\left[\frac{\mathrm{D}_0}{\rho^2}\cos 2\theta - 2\sum_{n=1}^{\infty}\frac{(2n-1)\,\mathrm{E}_{2n}}{\rho^{2n}}\cos 2n\theta + 2\sum_{n=1}^{\infty}\left\{\frac{n\,(2n+1)\,\mathrm{D}_{2n}}{\rho^{2n+2}}+\frac{n\,(2n-1)\,\mathrm{E}_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\,\theta\right]$$

$$= \frac{1}{b^2}\left[\frac{\mathrm{D}_0-2\,\mathrm{E}_2}{\rho^2}\cos 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)\,(n\mathrm{D}_{2n}-\mathrm{E}_{2n+2})}{\rho^{2n+2}} + \frac{n\,(2n-1)\,\mathrm{E}_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\,\theta\right]. \quad (12)$$

Similarly

$$\widehat{yy} = \frac{1}{b^2}\left[-\frac{\mathrm{D}_0+2\mathrm{E}_2}{\rho^2}\cos 2\theta - 2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)\,(n\mathrm{D}_{2n}+\mathrm{E}_{2n+2})}{\rho^{2n+2}} + \frac{n\,(2n-1)\,\mathrm{E}_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\,\theta\right], \quad (13)$$

$$\widehat{xy} = -\frac{1}{b^2}\left[\frac{\mathrm{D}_0}{\rho^2}\sin 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{n\,(2n+1)\,\mathrm{D}_{2n}}{\rho^{2n+2}}+\frac{n\,(2n-1)\,\mathrm{E}_{2n}}{\rho^{2n}}\right\}\sin(2n+2)\theta\right]. \quad (14)$$

We have now to construct χ_{2r+1} so that it produces on the edge of the strip stresses cancelling the values of $\widehat{yy}$ and $\widehat{xy}$ in (13) and (14). A method for constructing such a biharmonic function has been given by Filon* and was expressed in a modified and convenient form in a recent paper by the Author.† The relevant formulæ may be stated as follows :—

Let χ be the biharmonic function, finite and single-valued in the strip, which gives stresses

$$\widehat{yy} = \phi(\xi), \quad \widehat{xy} = \psi(\xi), \qquad \text{on} \quad \eta = 1\,;$$

$$\widehat{yy} = \phi(\xi), \quad \widehat{xy} = -\psi(\xi), \qquad \text{on} \quad \eta = -1\,;$$

$\phi(\xi)$ and $\psi(\xi)$ being both even functions ; then

$$\chi = \chi' + \chi'', \quad (15)$$

* 'Phil. Trans. Roy Soc.,' A, vol. 201, pp. 63–155 (1903).

† 'Roy. Soc. Proc.,' A, vol. 124, pp. 89–119 (1929).

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where

$$\left.\begin{aligned}\chi' &= \frac{4b^2}{\pi}\int_0^\infty \frac{u\eta s\mathrm{S}-(s+uc)\,\mathrm{C}}{u^2\,\Sigma}\cos u\xi\,.\,du\int_0^\infty \phi\,(w)\cos uw\,.\,dw\\ \chi'' &= \frac{4b^2}{\pi}\int_0^\infty \frac{\eta c\mathrm{S}-s\mathrm{C}}{u\,\Sigma}\cos u\xi\,.\,du\int_0^\infty \psi\,(w)\sin uw\,.\,dw\end{aligned}\right\}, \quad . \; . \; . \quad (16)$$

and the symbols have the meanings

$$\left.\begin{aligned} s &= \sinh u, \qquad c = \cosh u\\ \mathrm{S} &= \sinh \eta u, \quad \mathrm{C} = \cosh \eta u\\ \Sigma &= \sinh 2u + 2u\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

In order that χ may be identical with χ_{2r+1} we must give $\phi\,(\xi)$ and $\psi\,(\xi)$ the following values, derived from (13) and (14);

$$\left.\begin{aligned}\phi\,(\xi) &= \frac{1}{b^2}\left[\frac{2\mathrm{E}_2+\mathrm{D}_0}{1+\xi^2}\cos 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)\,(n\mathrm{D}_{2n}+\mathrm{E}_{2n+2})}{(1+\xi^2)^{n+1}}\right.\right.\\ &\qquad\qquad \left.\left. + \frac{n\,(2n-1)\,\mathrm{E}_{2n}}{(1+\xi^2)^n}\right\}\cos\,(2n+2)\,\theta\right]\\ \psi\,(\xi) &= \frac{1}{b^2}\left[\frac{\mathrm{D}_0}{1+\xi^2}\sin 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{n\,(2n+1)\,\mathrm{D}_{2n}}{(1+\xi^2)^{n+1}}\right.\right.\\ &\qquad\qquad \left.\left. + \frac{n\,(2n-1)\,\mathrm{E}_{2n}}{(1+\xi^2)^n}\right\}\sin\,(2n+2)\,\theta\right]\end{aligned}\right\} \quad . \; . \; . \; . \quad (18)$$

θ being the acute angle defined by the equation

$$\tan\theta = \xi. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

It is therefore necessary to evaluate definite integrals of the types

$$\int_0^\infty \frac{\cos 2n\theta\,.\,\cos uw}{(1+w^2)^n}\,.\,dw, \qquad \int_0^\infty \frac{\cos\,(2n+2)\,\theta\,.\,\cos uw}{(1+w^2)^n}\,.\,dw,$$

and the corresponding sine integrals, where now

$$\tan\theta = w.$$

To do this, consider first the complex integral

$$\int \frac{e^{iuw}}{(1+iw)^{2n}}\,.\,dw,$$

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taken round a contour consisting of the real axis and the upper half of a circle of large radius R with its centre at $w = 0$. The part due to the semi-circle tends to 0 as $R \to \infty$ and we are left with

$$\int_{-\infty}^{\infty} \frac{e^{iuw}}{(1+iw)^{2n}} \, . \, dw = \int_{-\infty}^{\infty} \frac{e^{i(uw-2n\theta)}}{(1+w^2)^n} \, . \, dw.$$

If w is replaced by $i + \alpha$ in the complex integral, the integrand becomes

$$e^{-u} \, . \, e^{iu\alpha}/(i\alpha)^{2n},$$

and the residue at the pole is clearly

$$\frac{e^{-u} \, . \, (iu)^{2n-1}}{i^{2n} \, . \, (2n-1)\,!} = \frac{e^{-u} \, . \, u^{2n-1}}{i \, . \, (2n-1)\,!}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{e^{i(uw-2n\theta)}}{(1+w^2)^n} \, dw = \frac{2\pi e^{-u} \, . \, u^{2n-1}}{(2n-1)\,!},$$

or

$$\int_{-\infty}^{\infty} \frac{\cos(uw-2n\theta)}{(1+w^2)^n} \, . \, dw = \frac{2\pi e^{-u} \, . \, u^{2n-1}}{(2n-1)\,!}.$$

Because of the relation between θ and w, the integrand is an even function, and the result may be written

$$\int_{0}^{\infty} \frac{\cos(uw-2n\theta)}{(1+w^2)^n} \, . \, dw = \frac{\pi e^{-u} \, . \, u^{2n-1}}{(2n-1)\,!}. \quad \quad (20)$$

Next take the corresponding complex integral

$$\int \frac{e^{-iuw}}{(1-iw)^{2n}} \, dw$$

round the same contour: the integrand has no pole within the contour and we obtain

$$\int_{0}^{\infty} \frac{\cos(uw+2n\theta)}{(1+w^2)^2} \, . \, dw = 0. \quad \quad (21)$$

By addition and subtraction of (20) and (21), we now have two of the required integrals expressed in the form

$$\int_{0}^{\infty} \frac{\cos 2n\theta \, . \, \cos uw}{(1+w^2)^n} \, . \, dw = \int_{0}^{\infty} \frac{\sin 2n\theta \, . \, \sin uw}{(1+w^2)^n} \, . \, dw = \frac{\pi \, e^{-u} \, . \, u^{2n-1}}{2 \, . \, (2n-1)!}. \quad \quad (22)$$

To obtain the value of the other pair of integrals, we replace the former complex integrals by the pair

$$\int \frac{(1+w^2)\, e^{iuw}}{(1+iw)^{2n+2}} \, . \, dw, \quad \int \frac{(1+w^2)\, e^{iuw}}{(1-iw)^{2n+2}} \, . \, dw,$$

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the contour being unchanged. When w is replaced by $i + \alpha$, the first integrand becomes

$$\frac{e^{-u} \cdot e^{iu\alpha}(\alpha^2 + 2i\alpha)}{(i\alpha)^{2n+2}}$$

and the residue is

$$\frac{e^{-u}}{i^{2n+2}}\left[\frac{(iu)^{2n-1}}{(2n-1)\,!} + 2i\frac{(iu)^{2n}}{(2n)\,!}\right] = \frac{e^{-u} \cdot u^{2n-1}}{i \cdot (2n)\,!} \cdot 2\,(u - n).$$

Hence, proceeding as before, we find

$$\int_0^\infty \frac{\cos\,(2n+2)\,\theta \cdot \cos uw}{(1+w^2)^n} \cdot dw = \int_0^\infty \frac{\sin\,(2n+2)\,\theta \cdot \sin uw}{(1+w^2)^n} \cdot dw$$

$$= \frac{\pi e^{-u} \cdot u^{2n-1}\,(u-n)}{(2n)\,!}. \quad . \quad . \quad . \quad . \quad (23)$$

It is now possible to write down the values of the integrals

$$\int_0^\infty \phi\,(w)\cos uw \cdot dw, \qquad \int_0^\infty \psi\,(w)\sin uw \cdot dw.$$

They are

$$\int_0^\infty \phi\,(w)\cos uw \cdot dw$$

$$= \frac{\pi\,e^{-u}}{b^2}\Bigg[(2E_2 + D_0)\cdot\frac{u}{2} + 2\sum_{n=1}^{\infty}(2n+1)\,(nD_{2n} + E_{2n+2})\cdot\frac{u^{2n+1}}{2\,.\,(2n+1)\,!}$$

$$+ 2\sum_{n=1}^{\infty} n\,(2n-1)\,E_{2n}\cdot\frac{u^{2n-1}\,(u-n)}{(2n)\,!}\Bigg]$$

$$= \frac{\pi\,e^{-u}\cdot u}{b^2}\left[\tfrac{1}{2}D_0 + \tfrac{1}{2}\sum_{n=1}^{\infty}(D_{2n} - E_{2n+2})\,\frac{u^{2n}}{(2n-1)\,!} + \sum_{n=1}^{\infty}E_{2n}\,\frac{u^{2n-1}}{(2n-2)\,!}\right], \quad . \quad . \quad . \quad (24)$$

$$\int_0^\infty \psi\,(w)\sin uw \cdot dw$$

$$= \frac{\pi\,e^{-u}}{b^2}\Bigg[D_0\cdot\frac{u}{2} + 2\sum_{n=1}^{\infty} n\,(2n+1)\,D_{2n}\cdot\frac{u^{2n+1}}{2\,.\,(2n+1)\,!}$$

$$+ 2\sum_{n=1}^{\infty} n\,(2n-1)\,E_{2n}\,.\,\frac{u^{2n-1}\,(u-n)}{(2n)\,!}\Bigg]$$

$$= \frac{\pi\,e^{-u}\cdot u}{b^2}\left[\frac{D_0 - 2E_2}{2} + \sum_{n=1}^{\infty}\frac{nD_{2n} - (n+1)\,E_{2n+2}}{(2n)\,!}\,u^{2n} + \sum_{n=1}^{\infty}\frac{E_{2n}}{(2n-2)\,!}\cdot u^{2n-1}\right]. \quad (25)$$

When these values are substituted into (16) the value of χ in (15) will be that of χ_{2r+1}. We have now to determine the stresses produced by this function at the cir-

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cumference of the hole, and for this purpose it is convenient to express χ_{2r+1} as a series in ascending powers of ρ. Writing

$$\left.\begin{aligned}\Phi &= \frac{b^2}{\pi u}\int_0^\infty \phi(w)\cos uw\,.\,dw\\ \Psi &= \frac{b^2}{\pi u}\int_0^\infty \psi(w)\sin uw\,.\,dw\end{aligned}\right\}, \qquad (26)$$

we have

$$\chi_{2r+1} = 4\int_0^\infty \frac{\eta S\cos u\xi}{\Sigma}(s\Phi + c\Psi)\,du - 4\int_0^\infty \frac{C\cos u\xi}{\Sigma}\left\{\frac{s+uc}{u}\Phi + s\Psi\right\}du. \qquad (27)$$

Now it is well known that

$$\begin{aligned}S\,.\cos u\xi &= \sinh(u\rho\cos\theta)\,.\cos(u\rho\sin\theta)\\ &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n+1}}{(2n+1)!}\cos(2n+1)\,\theta,\end{aligned}$$

and therefore

$$\begin{aligned}\eta S\cos u\xi &= \rho\cos\theta\sum_{n=0}^{\infty}\frac{(u\rho)^{2n+1}}{(2n+1)!}\cos(2n+1)\,\theta\\ &= \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{u^{2n+1}\rho^{2n+2}}{(2n+1)!}\{\cos 2n\theta + \cos(2n+2)\,\theta\}\\ &= \tfrac{1}{2}\left[u\rho^2 + \sum_{n=1}^{\infty}\left\{\frac{u^{2n-1}\rho^{2n}}{(2n-1)!} + \frac{u^{2n+1}\rho^{2n+2}}{(2n+1)!}\right\}\cos 2n\theta\right]. \qquad (28)\end{aligned}$$

Also

$$C\cos u\xi = \sum_{n=0}^{\infty}\frac{(u\rho)^{2n}}{(2n)!}\cos 2n\theta. \qquad (29)$$

Substituting these series into (27), we may now express χ_{2r+1} in the form

$$\chi_{2r+1} = \sum_{n=0}^{\infty}(L_{2n}^{(r)} + M_{2n}^{(r)}\rho^2)\,\rho^{2n}\cos 2n\theta, \qquad (30)$$

where

$$\left.\begin{aligned}L_0^{(r)} &= -4\int_0^\infty\{(s+uc)\,\Phi + us\Psi\}\frac{du}{u\Sigma}\\ L_{2n}^{(r)} &= \frac{4}{(2n)!}\int_0^\infty[s\{(n-1)\,\Phi - u\Psi\} + c\,(n\Psi - u\Phi)]\frac{u^{2n-1}}{\Sigma}\,du\\ M_{2n}^{(r)} &= \frac{2}{(2n+1)!}\int_0^\infty\frac{s\Phi + c\Psi}{\Sigma}u^{2n+1}\,.\,du\end{aligned}\right\}, \qquad (31)$$

the last equation applying also for $n = 0$.

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The integral for $L_0^{(r)}$ contains a divergent part* which could have been removed by making a slight modification in the formula for χ' in (16). But as the constant term in χ_{2r+1} is without effect on the stresses, it is trivial and will be supposed omitted. The other coefficients must now be expressed in a form suitable for calculation.

Taking $M_{2n}^{(r)}$ first, we have

$$2(s\Phi + c\Psi) = e^u(\Phi + \Psi) + e^{-u}(\Psi - \Phi)$$

$$= (D_0 - E_2) + \sum_{p=1}^{\infty}\{2p\,D_{2p} - (2p+1)\,E_{2p+2}\}\frac{u^{2p}}{(2p)!}$$

$$+ 2\sum_{p=1}^{\infty} E_{2p}\frac{u^{2p-1}}{(2p-2)!} - e^{-2u}\sum_{p=1}^{\infty} E_{2p}\frac{u^{2p-2}}{(2p-2)!}.$$

Substituting this into (31) and restoring the suffixes (r) to the coefficients of χ_{2r}, we now obtain

$$M_{2n}^{(r)} = \frac{1}{(2n+1)!}\Bigg[(D_0^{(r)} - E_2^{(r)})\,I_{2n+1} + \sum_{p=1}^{\infty}\{2p\,D_{2p}^{(r)} - (2p+1)\,E_{2p+2}^{(r)}\}\frac{I_{2n+2p+1}}{(2p)!}$$

$$+ 2\sum_{p=1}^{\infty} E_{2p}^{(r)}\frac{I_{2n+2p}}{(2p-2)!} - \sum_{p=1}^{\infty} E_{2p}^{(r)}\frac{J_{2n+2p-1}}{(2p-2)!}\Bigg],$$

the new symbols denoting the definite integrals

$$\left.\begin{aligned} I_s &= \int_0^{\infty}\frac{u^s}{\Sigma}\,du \\ J_s &= \int_0^{\infty}\frac{u^s}{\Sigma}\,e^{-2u}\,.\,du \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (32)$$

After a little re-arrangement of the terms, the equation for $M_{2n}^{(r)}$ becomes

$$M_{2n}^{(r)} = {}^n\gamma_0\,D_0^{(r)} + \sum_{p=1}^{\infty}\{{}^n\gamma_p\,D_{2p}^{(r)} + {}^n\delta_p\,E_{2p}^{(r)}\}, \quad \ldots \ldots \quad (33)$$

where

$$\left.\begin{aligned} {}^n\gamma_0 &= \frac{I_{2n+1}}{(2n+1)!} \\ {}^n\gamma_p &= \frac{I_{2n+2p+1}}{(2n+1)!\,(2p-1)!} \\ {}^n\delta_p &= -\frac{1}{(2n+1)!\,(2p-2)!}\{(2p-1)\,I_{2n+2p-1} - 2I_{2n+2p} + J_{2n+2p-1}\} \end{aligned}\right\} \quad \ldots \quad (34)$$

* A discussion of the convergence of these integrals and of the significance of divergent terms will be found in the Author's paper cited above.

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Turning now to $L_{2n}^{(r)}$, we have

$$2\left[s\{(n-1)\Phi-u\Psi\}+c(n\Psi-u\Phi)\right]$$
$$=e^{u}\{(n-u)(\Phi+\Psi)-\Phi\}-e^{-u}\{(n+u)(\Phi-\Psi)-\Phi\}$$
$$=(n-u)\left[(D_0-E_2)+\sum_{p=1}^{\infty}\{2pD_{2p}-(2p+1)E_{2p+2}\}\frac{u^{2p}}{(2p)!}+2\sum_{p=1}^{\infty}E_{2p}\frac{u^{2p-1}}{(2p-2)!}\right]$$
$$-(n+u)e^{-2u}\sum_{p=1}^{\infty}E_{2p}\frac{u^{2p-2}}{(2p-2)!}$$
$$-(1-e^{-2u})\left[\tfrac{1}{2}D_0+\tfrac{1}{2}\sum_{p=1}^{\infty}(D_{2p}-E_{2p+2})\frac{u^{2p}}{(2p-1)!}+\sum_{p=1}^{\infty}E_{2p}\frac{u^{2p-1}}{(2p-2)!}\right],$$

which may be re-arranged in the form

$$\tfrac{1}{2}\{(2n-1)-2u+e^{-2u}\}\left\{D_0+\sum_{p=1}^{\infty}\frac{u^{2p}}{(2p-1)!}D_{2p}\right\}$$
$$+\sum_{p=0}^{\infty}\{2(n+p)u^{2p+1}-(2np+n-p)u^{2p}-2u^{2p+2}-e^{-2u}(n+p)u^{2p}\}\frac{E_{2p+2}}{(2p)!}.$$

Substituting this into (31) and again restoring the suffixes, we find, after some further re-arrangement of the terms

$$L_{2n}^{(r)}={}^{n}\alpha_0D_0^{(r)}+\sum_{p=1}^{\infty}\{{}^{n}\alpha_p\,.\,D_{2p}^{(r)}+{}^{n}\beta_pE_{2p}^{(r)}\},\quad .\ .\ .\ .\ .\ .\ .\ .\quad (35)$$

where

$$\left.\begin{aligned}{}^{n}\alpha_0&=\frac{1}{(2n)!}\{(2n-1)I_{2n-1}-2I_{2n}+J_{2n-1}\}\\ {}^{n}\alpha_p&=\frac{1}{(2n)!\,(2p-1)!}\{(2n-1)I_{2n+2p-1}-2I_{2n+2p}+J_{2n+2p-1}\}\\ {}^{n}\beta_p&=\frac{2}{(2n)!\,(2p-2)!}\{2(n+p-1)I_{2(n+p-1)}-(2np-n-p+1)I_{2n+2p-3}\\ &\qquad-2I_{2n+2p-1}-(n+p-1)J_{2n+2p-3}\}\end{aligned}\right\}.\quad (36)$$

The coefficients in χ_{2r+1} are thus determined in terms of those of χ_{2r}. To complete the cycle we have now to determine the coefficients in χ_{2r+2} in terms of those of χ_{2r+1}. The stresses due to χ_{2r+1} are given immediately by the differentiation of (30) as

$$\left.\begin{aligned}\widehat{rr}&=\frac{2}{b^2}\left[M_0^{(r)}-\sum_{n=1}^{\infty}\{n(2n-1)L_{2n}^{(r)}+(n-1)(2n+1)M_{2n}^{(r)}\rho^2\}\rho^{2n-2}\cos 2n\theta\right]\\ \widehat{r\theta}&=\frac{2}{b^2}\sum_{n=1}^{\infty}\{n(2n-1)L_{2n}^{(r)}+n(2n+1)M_{2n}^{(r)}\rho^2\}\rho^{2n-2}\sin 2n\theta\\ \widehat{\theta\theta}&=\frac{2}{b^2}\left[M_0^{(r)}+\sum_{n=1}^{\infty}\{n(2n-1)L_{2n}^{(r)}+(n+1)(2n+1)M_{2n}^{(r)}\rho^2\}\rho^{2n-2}\cos 2n\theta\right]\end{aligned}\right\}.$$
(37)

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The first two of these must be cancelled at the edge of the hole by the stresses due to χ_{2r+2}. If we suppose this expanded in the same form as χ_{2r}, viz.—

$$\chi_{2r+2} = -\,D_0^{(r+1)}\log\rho + \sum_{n=1}^{\infty}\left\{\frac{D_{2n}^{(r+1)}}{\rho^{2n}} + \frac{E_{2n}^{(r+1)}}{\rho^{2n-2}}\right\}\cos 2n\theta, \quad . \quad . \quad . \quad . \quad (38)$$

the corresponding stresses will be given by adding the suffix $(r+1)$ to the coefficients in equations (8) and (9). Putting $\rho = \lambda$ and equating the coefficients to the negatives of those in (37), we obtain the following equations :—

$$\left.\begin{aligned}
&\frac{D_0^{(r+1)}}{\lambda^2} = 2M_0^{(r)} \\
&\frac{n\,(2n+1)}{\lambda^{2n+2}}\,D_{2n}^{(r+1)} + \frac{(n+1)\,(2n-1)}{\lambda^{2n}}\,E_{2n}^{(r+1)} = -\,\{n\,(2n-1)\,L_{2n}^{(r)} \\
&\qquad\qquad + (n-1)\,(2n+1)\,M_{2n}^{(r)}\lambda^2\}\;\lambda^{2n-2} \\
&\frac{n\,(2n+1)}{\lambda^{2n+2}}\,D_{2n}^{(r+1)} + \frac{n\,(2n-1)}{\lambda^{2n}}\,E_{2n}^{(r+1)} = \{n\,(2n-1)\,L_{2n}^{(r)} \\
&\qquad\qquad + n\,(2n+1)\,M_{2n}^{(r)}\lambda^2\}\;\lambda^{2n-2},
\end{aligned}\right\}$$

which may now be solved for $D_{2n}^{(r+1)}$ and $E_{2n}^{(r+1)}$ in the form

$$\left.\begin{aligned}
D_0^{(r+1)} &= 2\,M_0^{(r)}\lambda^2 \\
D_{2n}^{(r+1)} &= \lambda^{4n}\,\{(2n-1)\,L_{2n}^{(r)} + 2n\lambda^2 M_{2n}^{(r)}\} \\
E_{2n}^{(r+1)} &= -\,\lambda^{4n-2}\,\{2nL_{2n}^{(r)} + (2n+1)\,\lambda^2 M_{2n}^{(r)}\}
\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Equations (33) to (36) and (39) solve completely the problem of determining the coefficients of each approximation in terms of those of the preceding approximation. It remains, however, first to show that the expansion of χ_{2r+1} in powers of ρ was a legitimate operation, and secondly to determine the numerical values of the coefficients defined in (34) and (36).

Convergence of the Series Used in the Previous Section.

It is clear that the nature of the convergence or divergence of the series (30) for χ_{2r+1} will be determined by the magnitude of the coefficients in χ_{2r} and by the asymptotic behaviour of the integrals of the type I_s and J_s for large values of s. Consider the slightly more general integral

$$I_{s,t} = \int_0^\infty \frac{u^s e^{-2tu}}{\Sigma}\,du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

where $t > -1$ and is not necessarily an integer. In the notation of (32), we then have

$$I_s = I_{s,0}, \quad J_s = I_{s,1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

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so that the results of the present discussion may be utilised later in dealing with these special integrals.

To obtain an upper bound to the value of (40) we divide the range of integration into two parts, writing

$$I_{s,t} = \int_0^k \frac{u^s e^{-2tu}}{\Sigma} du + \int_k^\infty \frac{u^s e^{-2tu}}{\Sigma} du,$$

where k satisfies the equation

$$4k = e^{-2k}.$$

The numerical value of k is about $0 \cdot 18$.

Since

$$\begin{aligned}\Sigma &= \sinh 2u + 2u \\ &= \tfrac{1}{2}(e^{2u} - e^{-2u} + 4u),\end{aligned}$$

we have, for all values of u, $\Sigma > 4u$, and this inequality is good enough for the lower range $(0, k)$ of integration; but in the upper range we have $4u > e^{-2u}$ and can use the much stronger inequality $\Sigma > \frac{1}{2} e^{2u}$. Hence

$$\begin{aligned}I_{s,t} &< \tfrac{1}{4}\int_0^k u^{s-1} \,.\, e^{-2tu} du + 2\int_k^\infty u^s e^{-2(t+1)u} du \\ &< \tfrac{1}{4} k^{s-1} \int_0^k e^{-2tu} du + \frac{2}{\{2(t+1)\}^{s+1}} \int_0^\infty v^s e^{-v} dv \\ &= \tfrac{1}{4} k^{s-1} \,.\, \frac{1 - e^{-2kt}}{2t} + \frac{s\,!}{2^s (t+1)^{s+1}},\end{aligned}$$

i.e.,

$$I_{s,t} < \frac{s\,!}{2^s}\left[\frac{1}{(t+1)^{s+1}} + \frac{2^{s-3} k^{s-1}}{t \,.\, s\,!}\right]. \quad \text{. . . . . (42)}$$

This inequality will be of use in the next section. At present we require it in the less explicit form

$$I_{s,t} < \frac{s\,!}{2^s} \,.\, \frac{1}{(t+1)^{s+1}} (1 + \varepsilon_s) \quad \text{. . . . . (43)}$$

where $\varepsilon_s \to 0$ as $s \to \infty$.

The sign of inequality in (43) may, however, be replaced by one of equality. For we have, evidently,

$$\int_0^\infty \frac{e^{-2tu} \,.\, u^s (e^{2u} - e^{-2u} + 4u)}{\Sigma} du = 2\int_0^\infty u^s e^{-2tu} du$$

or

$$I_{s,t-1} - I_{s,t+1} + 4I_{s+1,t} = \frac{s\,!}{2^s \,.\, t^{s+1}}.$$

Changing t to $t + 1$ and transposing,

$$I_{s,t} = \frac{s\,!}{2^s (t+1)^{s+1}} + I_{s,t+2} - 4I_{s+1,t+1}. \quad \text{. . . . . (44)}$$

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If we now apply (43) to the two integrals on the right of (44) and write ε_s' for the larger of the two values of ε_s, we see that

$$I_{s,t} = \frac{s\,!}{2^s\,(t+1)^{s+1}}\,(1+\eta_s),$$

where

$$|\eta_s| < \left\{\left(\frac{t+1}{t+3}\right)^{s+1} + 2\left(\frac{s+1}{t+2}\right)\left(\frac{t+1}{t+2}\right)^{s+1}\right\}(1+\varepsilon_s'),$$

and tends to 0 as s tends to ∞. Hence we have the asymptotic formula

$$I_{s,t} \sim \frac{s\,!}{2^s\,(t+1)^{s+1}} \quad \dots\dots\dots\dots \quad (45)$$

The above method of establishing (45) is a little artificial but has the advantage that it gives two results, viz., (42) and (44), which will be of great value in the calculation of the integrals I_s and J_s.

We have next to make some assumption about the magnitude of the coefficients in the series for χ_{2r}. This series must converge outside a circle of radius $\lambda' \leq \lambda$. A sufficient condition for this is

$$\left.\begin{array}{l} |D_{2n}^{(r)}| < k\lambda'^{2n} \\ |E_{2n}^{(r)}| < k\lambda'^{2n-2} \end{array}\right\}, \quad \dots\dots\dots\dots \quad (46)$$

k being a constant. From (24) and (26) we have

$$\Phi = e^{-u}\left[\tfrac{1}{2}D_0 + \tfrac{1}{2}\sum_{n=1}^{\infty}(D_{2n} - E_{2n+2})\frac{u^{2n}}{(2n-1)\,!} + \sum_{n=1}^{\infty} E_{2n}\frac{u^{2n-1}}{(2n-2)\,!}\right].$$

Hence

$$|\Phi| < ke^{-u}\left[\tfrac{1}{2} + \sum_{n=1}^{\infty}\frac{(\lambda' u)^{2n}}{(2n-1)\,!} + \sum_{n=1}^{\infty} u\,\frac{(\lambda' u)^{2n-1}}{(2n-2)\,!}\right]$$

$$= ke^{-u}\left[\tfrac{1}{2} + \lambda' u \sinh \lambda' u + u \cosh \lambda' u\right].$$

Similarly

$$|\Psi| < ke^{-u}\left[\tfrac{1}{2} + \lambda' u \sinh \lambda' u + (u+1)\cosh \lambda' u\right].$$

Hence, using (31),

$$|M_{2n}^{(r)}| \leq \frac{2}{(2n+1)\,!}\int_0^\infty \frac{u^{2n+1}}{\Sigma}\,(s\,|\,\Phi\,| + c\,|\,\Psi\,|)\,du$$

$$< \frac{2k}{(2n+1)\,!}\int_0^\infty \frac{u^{2n+1}}{\Sigma}\,.\,P\,du, \quad \dots\dots\dots\dots \quad (47)$$

where

$$P = \tfrac{1}{2} + \lambda' u \sinh \lambda' u + (u + \tfrac{1}{2})\cosh \lambda' u + \tfrac{1}{2}e^{-2u}\,.\cosh \lambda' u,$$

$$= \tfrac{1}{4}\{2 + e^{\lambda' u}(1 + \overline{2+\lambda'}\,.\,u) + e^{-\lambda' u}(1 + \overline{2-\lambda'}u) + e^{-(2-\lambda')u} + e^{-(2+\lambda')u}\}.$$

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It is clear from (45) that, when this is substituted in (47), the important term when n is large will be

$$\tfrac{1}{4}(2+\lambda')\,u\,e^{\lambda' u},$$

and we have

$$\begin{aligned}|\mathrm{M}_{2n}^{(r)}| &< \frac{2k}{(2n+1)\,!}\int_0^\infty \frac{u^{2n+1}}{\Sigma}\,.\,\mathrm{P}\,du \\ &\sim \frac{k\,(2+\lambda')}{2\,.\,(2n+1)\,!}\int_0^\infty \frac{u^{2n+2}}{\Sigma}\,e^{\lambda' u}\,du \\ &\sim \frac{k\,(2+\lambda')}{2\,.\,(2n+1)\,!}\,\frac{(2n+1)\,!}{2^{2n+1}(1-\tfrac{1}{2}\lambda')^{2n+2}} \\ &= \frac{k\,(2+\lambda')}{(2-\lambda')^{2n+2}}.\end{aligned}$$

The discussion of $\mathrm{L}_{2n}^{(r)}$ is a little longer: it leads to a result of the same kind. It is clear therefore that the series for χ_{2r+1} will converge at least in a circle of radius $2-\lambda'$. If it diverges outside this circle the continuation of the function will be given by the integral formula (27), from which expansions about other centres may be developed.

It is clear also that when the above inequalities for $|\mathrm{M}_{2n}^{(r)}|$ and $|\mathrm{L}_{2n}^{(r)}|$ are substituted into (39) the resulting inequalities for $\mathrm{D}_{2n}^{(r+1)}$ and $\mathrm{E}_{2n}^{(r+1)}$ will be at least as strong as those assumed in (46), so that the subsequent series will all converge in regions not smaller than the circle of convergence of χ_1.

In the tension problem the series for χ_0 is finite. It follows that the χ_{2r+1} series will converge in a circle of radius at least 2. The foregoing argument does not show that the radius of convergence cannot be greater than this, but it will be seen later that the series actually obtained converges rather slowly when ρ exceeds 1.

Evaluation of the Integrals.

The coefficients ${}^n\alpha_p$, ${}^n\beta_p$, ${}^n\gamma_p$, ${}^n\delta_p$, depend in a rather involved way on the integrals I_s and J_s. Moreover, these coefficients enter afresh at each cycle of the approximation. It follows that errors in the values of I_s and J_s will tend to multiply themselves as the calculation proceeds, and it is desirable that their values should be known with some degree of accuracy. The values of I_s have therefore been calculated, for values of s from 1 to 20, to 5 significant figures. Less accuracy is needed in the values of J_s except for small values of s. The method of calculation is different according to the magnitude of s. When s is fairly large, the integrals are easily found from asymptotic formulæ derivable from equation (44) of the last section. For smaller values of s they are best found by quadratures.

Putting $t=0$ in (44) we obtain

$$\mathrm{I}_s = \frac{s\,!}{2^s} + \mathrm{I}_{s,2} - 4\mathrm{I}_{s+1,1}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

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Again using (44) in the last term of (48), the equation becomes

$$\mathrm{I}_s = \frac{s\,!}{2^s} - \frac{(s+1)\,!}{2^{2s+1}} + \mathrm{I}_{s,2} - 4\,\mathrm{I}_{s+1,3} + 16\,\mathrm{I}_{s+2,2}. \quad \ldots \ldots \quad (49)$$

When s is large the integrals on the right hand of equations (48) and (49), though large compared with unity, are small by comparison with the integrated terms. An upper limit to their value is easily found by using (42). It is then seen that, in (48), the first term alone will give the value of the integral correct to 5 significant figures if s exceeds 20, while in (49) the remainder terms may be neglected if s exceeds 14.

Proceeding in this way, we obtain the following series of asymptotic formulæ :—

$$\begin{aligned}
\mathrm{I}_s &= \frac{s\,!}{2^s}(1+\varepsilon_1) \\
&= \frac{s\,!}{2^s}\left(1 - \frac{s+1}{2^{s+1}} + \varepsilon_2\right) \\
&= \frac{s\,!}{2^s}\left\{\left(1+\frac{1}{3^{s+1}}\right) - \frac{s+1}{2^{s+1}} + \frac{4\,[s+1]_2}{3^{s+3}} + \varepsilon_3\right\} \\
&= \frac{s\,!}{2^s}\left\{\left(1+\frac{1}{3^{s+1}}\right) - 2\,[s+1]\left(\frac{1}{2^{s+2}}+\frac{2}{4^{s+2}}\right) + \frac{4\,[s+1]_2}{3^{s+3}} - \frac{8\,[s+1]_3}{4^{s+4}} + \varepsilon_4\right\} \\
&= \frac{s\,!}{2^s}\left\{\left(1+\frac{1}{3^{s+1}}+\frac{1}{5^{s+1}}\right) - 2\,[s+1]\left(\frac{1}{2^{s+2}}+\frac{2}{4^{s+2}}\right) + 4\,[s+1]_2\left(\frac{1}{3^{s+3}}+\frac{3}{5^{s+3}}\right)\right. \\
&\qquad\qquad \left. - \frac{8\,[s+1]_3}{4^{s+4}} + \frac{16\,[s+1]_4}{5^{s+5}} + \varepsilon_5\right\} \\
&= \frac{s\,!}{2^s}\left\{\left(1+\frac{1}{3^{s+1}}+\frac{1}{5^{s+1}}\right) - 2\,[s+1]\left(\frac{1}{2^{s+2}}+\frac{2}{4^{s+2}}+\frac{3}{6^{s+2}}\right) + 4\,[s+1]_2\left(\frac{1}{3^{s+3}}+\frac{3}{5^{s+3}}\right)\right. \\
&\qquad\qquad \left. - 8\,[s+1]_3\left(\frac{1}{4^{s+4}}+\frac{4}{6^{s+4}}\right) + \frac{16\,[s+1]_4}{5^{s+5}} - \frac{32\,[s+1]_5}{6^{s+6}} + \varepsilon_6\right\} \\
&= \frac{s\,!}{2^s}\left[\left(1+\frac{1}{3^{s+1}}+\frac{1}{5^{s+1}}+\frac{1}{7^{s+1}}\right) - 2\,[s+1]\left(\frac{1}{2^{s+2}}+\frac{2}{4^{s+2}}+\frac{3}{6^{s+2}}\right)\right. \\
&\qquad + 4\,[s+1]_2\left(\frac{1}{3^{s+3}}+\frac{3}{5^{s+3}}+\frac{6}{7^{s+3}}\right) - 8\,[s+1]_3\left(\frac{1}{4^{s+4}}+\frac{4}{6^{s+4}}\right) \\
&\qquad \left. + 16\,[s+1]_4\left(\frac{1}{5^{s+5}}+\frac{5}{7^{s+5}}\right) - \frac{32\,[s+1]_5}{6^{s+6}} + \frac{64\,[s+1]_6}{7^{s+7}} + \varepsilon_7\right], \quad . \; . \quad (50)
\end{aligned}$$

in which $[s+1]_p$ denotes the product of p factors starting with $s+1$ and ascending by integers, *i.e.*,

$$[s+1]_p = \frac{(s+p)\,!}{s\,!} \quad \ldots \ldots \ldots \ldots \quad (51)$$

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The remainders become rather long as the process is continued. Thus

$$\frac{s\,!}{2^s}\varepsilon_7 = \mathrm{I}_{s,8} - 16\,\mathrm{I}_{s+1,7} + 96\,\mathrm{I}_{s+2,8} - 640\,\mathrm{I}_{s+3,7} + 1280\,\mathrm{I}_{s+4,8} - 6144\,\mathrm{I}_{s+5,7} + 4096\,\mathrm{I}_{s+6,8} - 16384\,\mathrm{I}_{s+7,7},$$

but only the last term makes an important contribution. An upper limit to each remainder may be found from (42), which shows that

$$|\varepsilon_1| < 10^{-5} \quad \text{if} \quad s > 20$$
$$|\varepsilon_2| < 10^{-5} \quad \text{if} \quad s > 14$$
$$|\varepsilon_3| < 10^{-5} \quad \text{if} \quad s > 12$$
$$|\varepsilon_4| < 10^{-5} \quad \text{if} \quad s > 11$$
$$|\varepsilon_5| < 10^{-5} \quad \text{if} \quad s > 10$$
$$|\varepsilon_6| < 10^{-5} \quad \text{if} \quad s > 9$$
$$|\varepsilon_7| < 10^{-5} \quad \text{if} \quad s > 9$$

If the approximations are carried further the improvement is slow and, for smaller values of s, the method of calculation is impracticable. It is clear, however, that the approximations are leading to a double Dirichlet series in which the coefficients of the successive single series are the figurate numbers of successive orders. This interesting series may be obtained by a more direct method of expansion which is, however, less convenient for the examination of the remainder terms.

For the calculation of J_s the above method is less favourable, but, fortunately, when s exceeds 8 it is unnecessary to obtain the values of J_s to more than two significant figures, and for values of s much larger than this it ceases to affect the coefficients in any of the figures retained. The successive approximation formulæ are

$$\mathrm{J}_s = \frac{s\,!}{2^{2s+1}}(1 + \varepsilon_1')$$

$$= \frac{s\,!}{2^{2s+1}}\left\{\frac{1}{2^{s+1}} + \frac{1}{4^{s+1}} - \frac{2\,(s+1)}{3^{s+2}} + \varepsilon_2'\right\}. \qquad (52)$$

the second of these corresponding to the third for I_s, the intermediate one being omitted. The second equation will give the first two significant figures if s exceeds 10; the first gives an error of more than 1 per cent. until s approaches 20.

For the computation of the integrals for small values of s, $1/\Sigma$ was first tabulated to 9 decimal places for values of u from 0 to 6 at intervals of 0·25, and the values of u^s/Σ were then found for values of s from 1 to 11. To obtain the integrands of J_s those of I_s were then multiplied by values of e^{-2u}, correct to 8 decimal places.* Tables of differences of several of the integrands were then constructed, and GREGORY'S† formula was

* The tables used were NEWMAN'S, 'Camb. Phil. Soc. Trans.,' vol. 13, pp. 145–241 (1883).

† WHITTAKER and ROBINSON, "The Calculus of Observations," § 72.

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applied in order that the effects of the differences on the value of the integral might be observed. It was found that even fifth differences might affect the sixth figure in I_s or the fifth figure in J_s. The integrals were then all evaluated by applying WEDDLE'S Rule,* which is correct to fifth differences, to each quarter of the range. A few were checked by applying to each one-third of the range the NEWTON-COTES† formula, with eight intervals. This is correct to seventh differences. In the I_s integrals so tested there was always agreement to the fifth significant figure. All those integrals not otherwise tested were recalculated by SIMPSON'S formula, which always showed agreement in the first four figures and sometimes in the fifth.

The ranges of the integrals above $u = 6$ were calculated by replacing Σ by $\frac{1}{2}e^{2u}$ and evaluating the resulting elementary integral. It is easy to find an upper limit to the error introduced in this way and to verify that the fifth figure is never affected.

When s exceeds 8 the above method of calculation cannot give more than four significant figures correctly unless $1/\Sigma$ is tabulated, for values of u between 5 and 6, to more than 9 decimal places. The former method of calculation then becomes the more reliable. For $s = 9$, 10 and 11, both methods were used, agreement being reached in the first 4 figures.

The values of the integrals and their ratios to the asymptotic values, $s!/2^s$ of I_s are given in Table I. When s exceeds 20 it is sufficient to take the asymptotic value of I_s.

TABLE I.—Values of the Integrals I_s and J_s.

s.	I_s.	$2^sI_s/s!$	J_s.	$2^sJ_s/s!$
1	$3\cdot8429 \times 10^{-1}$	0·76858	$1\cdot1006 \times 10^{-1}$	0·22012
2	$3\cdot8392 \times 10^{-1}$	0·76784	$4\cdot396 \times 10^{-2}$	0·08792
3	$6\cdot2078 \times 10^{-1}$	0·82771	$3\cdot252 \times 10^{-2}$	0·04336
4	1·3253	0·88353	$3\cdot386 \times 10^{-2}$	0·02257
5	3·4706	0·92549	$4\cdot472 \times 10^{-2}$	0·01192
6	$1\cdot0735 \times 10^{1}$	0·95422	$7\cdot074 \times 10^{-2}$	0·00629
7	$3\cdot8302 \times 10^{1}$	0·97275	$1\cdot297 \times 10^{-1}$	0·00329
8	$1\cdot5504 \times 10^{2}$	0·98439	$2\cdot698 \times 10^{-1}$	0·00171
9	$7\cdot0233 \times 10^{2}$	0·99095	$6\cdot27 \times 10^{-1}$	0·00088
10	$3\cdot5257 \times 10^{3}$	0·99492	1·61	0·00045
11	$1\cdot9438 \times 10^{4}$	0·99718	4·5	0·00023
12	$1\cdot1676 \times 10^{5}$	0·99846	$1\cdot4 \times 10^{1}$	0·00012
13	$7\cdot5950 \times 10^{5}$	0·99916	$4\cdot5 \times 10^{1}$	0·00006
14	$5\cdot3234 \times 10^{6}$	0·99955	$1\cdot6 \times 10^{2}$	0·00003
15	$3\cdot9897 \times 10^{7}$	0·99976	$6\cdot0 \times 10^{2}$	0·00002
16	$3\cdot1921 \times 10^{8}$	0·99987	$2\cdot4 \times 10^{3}$	0·00001
17	$2\cdot7135 \times 10^{9}$	0·99993	$1\cdot0 \times 10^{4}$	—
18	$2\cdot4422 \times 10^{10}$	0·99996	$4\cdot5 \times 10^{4}$	—
19	$2\cdot3202 \times 10^{11}$	0·99999	2×10^{5}	—
20	$2\cdot3202 \times 10^{12}$	0·99999	1×10^{6}	—

* *Ibid.* § 75.

† *Ibid.* § 76.

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Evaluation of the Coefficients.

The coefficients of the transformation may now be calculated in a direct manner from equations (34) and (36). It is clear that these coefficients are not independent. The following relations between them are easy to verify :

$$\left.\begin{aligned}
&n\,.\,{}^{n}\alpha_p - p\,.\,{}^{p}\alpha_n = (n-p)\,.\,{}^{n-1}\gamma_p\,;\\
&p\,.\,{}^{n-1}\delta_{p+1} - n\,.\,{}^{p-1}\delta_{n+1} = (n-p)\,.\,{}^{n-1}\gamma_p\,;\\
&2n\,.\,{}^{n}\alpha_0 + {}^{n-1}\delta_1 = 2(n-1)\,.\,{}^{n-1}\gamma_0\,;\\
&{}^{n}\alpha_p + {}^{p-1}\delta_{n+1} = -\frac{1}{n}\,.\,{}^{n-1}\gamma_p\,;\\
&np\,({}^{n}\beta_{p+1} - {}^{p}\beta_{n+1}) = (p-n)\,.\,{}^{n-1}\gamma_p\,;\\
&n\,(2n-1)\,.\,{}^{n}\beta_p = p\,(2p-1)\,{}^{p}\beta_n\,;\\
&{}^{n-1}\gamma_p = {}^{p-1}\gamma_n\,;\\
&{}^{n}\alpha_p - {}^{n-1}\delta_{p+1} + {}^{n}\beta_{p+1} = {}^{n-1}\gamma_p - \frac{2(2n+1)}{p}\,.\,{}^{n}\gamma_p\,;\\
&{}^{n}\beta_1 = {}^{n-1}\delta_1 - 4\,(2n+1)\,{}^{n}\gamma_0.
\end{aligned}\right\} \quad \ldots\ldots \quad (53)$$

These nine equations have been selected in such a way that every coefficient finds a place in them for some values of the prefixes and suffixes. The coefficients were then all evaluated independently and (53) were used in a systematic manner to test every figure arrived at. In consequence it may be stated that important errors are most unlikely to remain. The four sets of coefficients are given in Tables II to V.

TABLE II.—Values of $-\,{}^{n}\alpha_p$.

	$n = 1.$	$n = 2.$	$n = 3.$	$n = 4.$	$n = 5.$	$n = 6.$	$n = 7.$
$p = 0$	0·13674	0·031489	0·005656	0·001038	0·000201	0·000041	$8{\cdot}9 \times 10^{-6}$
$p = 1$	0·99865	0·45889	0·16450	0·052939	0·016141	0·004786	0·001374
$p = 2$	1·4962	1·3544	0·81924	0·40283	0·17503	0·06944	0·02594
$p = 3$	1·1319	1·7166	1·5778	1·1016	0·64151	0·33048	0·15524
$p = 4$	0·6298	1·4485	1·8875	1·7673	1·3354	0·86497	—
$p = 5$	0·2950	0·9609	1·6801	2·0401	1·9382	—	—
$p = 6$	0·1238	0·5415	1·2274	1·8741	—	—	—

TABLE III.—Values of $-\,{}^{n}\beta_p$.

	$n = 1.$	$n = 2.$	$n = 3.$	$n = 4.$	$n = 5.$	$n = 6.$	$n = 7.$
$p = 1$	0·96807	0·24555	0·063167	0·015777	0·003932	0·000979	0·000243
$p = 2$	1·4733	1·2367	0·65456	0·28145	0·10748	0·03795	0·01282
$p = 3$	0·94750	1·6364	1·4759	0·96677	0·52208	0·24995	0·10896
$p = 4$	0·44176	1·3134	1·8048	1·6709	1·2220	0·75544	0·41400
$p = 5$	0·17693	0·80607	1·5662	1·9638	1·8548	1·4416	—
$p = 6$	0·06463	0·41742	1·0998	1·7807	2·1143	—	—

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TABLE IV.—Values of ${}^{n}\gamma_p$.

	$n = 0.$	$n = 1.$	$n = 2.$	$n = 3.$	$n = 4.$	$n = 5.$	$n = 6.$
$p = 0$	0·38429	0·10346	0·028922	0·007599	0·001935	0·000487	0·000122
$p = 1$	0·62078	0·57843	0·31918	0·13935	0·053566	0·019027	0·006407
$p = 2$	0·57843	1·0639	0·97546	0·64279	0·34883	0·16658	0·072627
$p = 3$	0·31918	0·97546	1·3499	1·2558	0·91621	0·56649	0·31050
$p = 4$	0·13935	0·64279	1·2558	1·5706	1·4837	1·1533	—
$p = 5$	0·053566	0·34883	0·91621	1·4837	1·7620	—	—
$p = 6$	0·019027	0·16658	0·56649	1·1533	—	—	—

TABLE V.—Values of ${}^{n}\delta_p$.

	$n = 0.$	$n = 1.$	$n = 2$	$n = 3.$	$n = 4.$	$n = 5.$	$n = 6.$
$p = 1$	0·27349	0·33288	0·14962	0·053899	0·017495	0·005363	0·001588
$p = 2$	0·3779	0·9178	0·81267	0·49046	0·24142	0·10482	0·041652
$p = 3$	0·1697	0·8225	1·2289	1·1271	0·78645	0·45818	0·23605
$p = 4$	0·05811	0·4941	1·1279	1·4689	1·3746	1·0386	0·67276
$p = 5$	0·01810	0·2421	0·78772	1·3747	1·6692	1·5858	—
$p = 6$	0·00543	0·1053	0·45826	1·0386	1·5858	—	—

The fifth significant figure, where it is given in these Tables, is not quite reliable. Wherever the error in it was likely to exceed 2 it has been omitted.

The Tension Problem.

To obtain the solution for a tension T applied to the strip, the rim of the hole as well as the edges of the strip being free from stress, we start with

$$\left.\begin{aligned} \chi_0' &= \tfrac{1}{4}b^2\mathrm{T}\rho^2\,(1+\cos 2\theta) \\ \chi_0 &= \tfrac{1}{4}b^2\mathrm{T}\left[-2\lambda^2\log\rho - \left(2\lambda^2 - \frac{\lambda^4}{\rho^2}\right)\cos 2\theta\right] \end{aligned}\right\}, \quad \ldots\ldots \quad (54)$$

the corresponding stresses being

$$\left.\begin{aligned} \widehat{rr}_0 &= \tfrac{1}{2}\mathrm{T}\left[\left(1-\frac{\lambda^2}{\rho^2}\right)-\left(1-4\frac{\lambda^2}{\rho^2}+3\frac{\lambda^4}{\rho^4}\right)\cos 2\theta\right] \\ \widehat{\theta\theta}_0 &= \tfrac{1}{2}\mathrm{T}\left[\left(1+\frac{\lambda^2}{\rho^2}\right)+\left(1+3\frac{\lambda^4}{\rho^4}\right)\cos 2\theta\right] \\ \widehat{r\theta}_0 &= \tfrac{1}{2}\mathrm{T}\left(1+2\frac{\lambda^2}{\rho^2}-3\frac{\lambda^4}{\rho^4}\right)\sin 2\theta \end{aligned}\right\}. \quad \ldots\ldots \quad (55)$$

We now have

$$\left.\begin{aligned} \mathrm{D}_0^{(0)} &= \tfrac{1}{2}b^2\mathrm{T}\lambda^2, \qquad \mathrm{D}_2^{(0)} = \tfrac{1}{4}b^2\mathrm{T}\lambda^3, \\ \mathrm{E}_2^{(0)} &= -\tfrac{1}{2}b^2\mathrm{T}\lambda^2, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (56)$$

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all the other coefficients being zero, and (33) and (35) now give

$$\left.\begin{aligned} L_{2n}^{(0)} &= b^2T\lambda^2\left\{\tfrac{1}{2}\left({}^n\alpha_0 - {}^n\beta_1\right) + \tfrac{1}{4}\,{}^n\alpha_1\lambda^2\right\} \\ M_{2n}^{(0)} &= b^2T\lambda^2\left\{\tfrac{1}{2}\left({}^n\gamma_0 - {}^n\delta_1\right) + \tfrac{1}{4}\,{}^n\gamma_1\lambda^2\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (57)$$

From these it is easy to calculate the values of the coefficients of χ_1 and to proceed to those of χ_2, retaining λ as a parameter. When, however, the approximation is to be carried further, it is best to give λ a definite numerical value. The calculation has been carried out for $\lambda = 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3,\ 0{\cdot}4,\ 0{\cdot}5$ and the coefficients are recorded in Tables VI to IX, where the following abbreviations have been used :

$$\left.\begin{aligned} d_0^{(r)} &= D_0^{(r)}/b^2T, & d_{2n}^{(r)} &= D_{2n}^{(r)}/b^2T, \\ e_{2n}^{(r)} &= E_{2n}^{(r)}/b^2T, & l_{2n}^{(r)} &= L_{2n}^{(r)}/b^2T, \\ m_0^{(r)} &= M_0^{(r)}/b^2T, & m_{2n}^{(r)} &= M_{2n}^{(r)}/b^2T. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (58)$$

TABLE VI.—Coefficients for $\lambda = 0{\cdot}1$ and $\lambda = 0{\cdot}2$.

	$\lambda = 0{\cdot}1$.		$\lambda = 0{\cdot}2$.	
	$r = 0$.	$r = 1$.	$r = 0$.	$r = 1$.
$d_0^{(r)}$	5×10^{-3}	$1{\cdot}14 \times 10^{-5}$	2×10^{-2}	$1{\cdot}97 \times 10^{-4}$
$d_2^{(r)}$	$2{\cdot}5 \times 10^{-5}$	$4{\cdot}11 \times 10^{-7}$	4×10^{-4}	$2{\cdot}54 \times 10^{-5}$
$d_4^{(r)}$	—	$3{\cdot}15 \times 10^{-11}$	—	$3{\cdot}05 \times 10^{-8}$
$d_6^{(r)}$	—	$1{\cdot}40 \times 10^{-15}$	—	$2{\cdot}13 \times 10^{-11}$
$d_8^{(r)}$	—	$5{\cdot}00 \times 10^{-20}$	—	$1{\cdot}19 \times 10^{-14}$
$e_2^{(r)}$	-5×10^{-3}	$-8{\cdot}23 \times 10^{-5}$	-2×10^{-2}	$-1{\cdot}28 \times 10^{-3}$
$e_4^{(r)}$	—	$-4{\cdot}21 \times 10^{-9}$	—	$-1{\cdot}02 \times 10^{-6}$
$e_6^{(r)}$	—	$-1{\cdot}68 \times 10^{-13}$	—	$-6{\cdot}41 \times 10^{-10}$
$e_8^{(r)}$	—	$-5{\cdot}72 \times 10^{-18}$	—	$-3{\cdot}41 \times 10^{-13}$
$l_2^{(r)}$	$4{\cdot}132 \times 10^{-3}$	—	$1{\cdot}623 \times 10^{-2}$	—
$l_4^{(r)}$	$1{\cdot}059 \times 10^{-3}$	—	$4{\cdot}098 \times 10^{-3}$	—
$l_6^{(r)}$	$2{\cdot}834 \times 10^{-4}$	—	$1{\cdot}084 \times 10^{-3}$	—
$l_8^{(r)}$	$7{\cdot}238 \times 10^{-5}$	—	$2{\cdot}736 \times 10^{-4}$	—
$l_{10}^{(r)}$	$1{\cdot}825 \times 10^{-5}$	—	$6{\cdot}816 \times 10^{-5}$	—
$l_{12}^{(r)}$	$4{\cdot}65 \times 10^{-6}$	—	$1{\cdot}80 \times 10^{-5}$	—
$l_{14}^{(r)}$	$1{\cdot}15 \times 10^{-6}$	—	$4{\cdot}41 \times 10^{-6}$	—
$m_0^{(r)}$	$5{\cdot}695 \times 10^{-4}$	—	$2{\cdot}464 \times 10^{-3}$	—
$m_2^{(r)}$	$-1{\cdot}133 \times 10^{-3}$	—	$-4{\cdot}353 \times 10^{-3}$	—
$m_4^{(r)}$	$-5{\cdot}955 \times 10^{-4}$	—	$-2{\cdot}286 \times 10^{-3}$	—
$m_6^{(r)}$	$-2{\cdot}280 \times 10^{-4}$	—	$-8{\cdot}703 \times 10^{-4}$	—
$m_8^{(r)}$	$-7{\cdot}646 \times 10^{-5}$	—	$-2{\cdot}898 \times 10^{-4}$	—
$m_{10}^{(r)}$	$-2{\cdot}390 \times 10^{-5}$	—	$-8{\cdot}99 \times 10^{-5}$	—
$m_{12}^{(r)}$	$-7{\cdot}30 \times 10^{-6}$	—	$-2{\cdot}86 \times 10^{-5}$	—

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TABLE VII.—Coefficients for $\lambda = 0 \cdot 3$.

	$r = 0$.	$r = 1$.	$r = 2$.
$d_0^{(r)}$	$4 \cdot 5 \times 10^{-2}$	$1 \cdot 124 \times 10^{-3}$	$-1 \cdot 96 \times 10^{-4}$
$d_2^{(r)}$	$2 \cdot 025 \times 10^{-3}$	$2 \cdot 733 \times 10^{-4}$	$4 \cdot 24 \times 10^{-5}$
$d_4^{(r)}$	—	$1 \cdot 600 \times 10^{-6}$	$2 \cdot 51 \times 10^{-7}$
$d_6^{(r)}$	—	$5 \cdot 475 \times 10^{-9}$	$8 \cdot 49 \times 10^{-10}$
$d_8^{(r)}$	—	$1 \cdot 49 \times 10^{-11}$	$2 \cdot 34 \times 10^{-12}$
$e_2^{(r)}$	$-4 \cdot 5 \times 10^{-2}$	$-6 \cdot 148 \times 10^{-3}$	$-9 \cdot 57 \times 10^{-4}$
$e_4^{(r)}$	—	$-2 \cdot 381 \times 10^{-5}$	$-3 \cdot 74 \times 10^{-6}$
$e_6^{(r)}$	—	$-7 \cdot 319 \times 10^{-8}$	$-1 \cdot 135 \times 10^{-8}$
$e_8^{(r)}$	—	$-1 \cdot 899 \times 10^{-10}$	$-2 \cdot 97 \times 10^{-11}$
$l_2^{(r)}$	$3 \cdot 539 \times 10^{-2}$	$5 \cdot 558 \times 10^{-3}$	—
$l_4^{(r)}$	$8 \cdot 704 \times 10^{-3}$	$1 \cdot 376 \times 10^{-3}$	—
$l_6^{(r)}$	$2 \cdot 255 \times 10^{-3}$	$3 \cdot 514 \times 10^{-4}$	—
$l_8^{(r)}$	$5 \cdot 561 \times 10^{-4}$	$8 \cdot 748 \times 10^{-5}$	—
$l_{10}^{(r)}$	$1 \cdot 352 \times 10^{-4}$	$2 \cdot 185 \times 10^{-5}$	—
$l_{12}^{(r)}$	$3 \cdot 253 \times 10^{-5}$	$5 \cdot 476 \times 10^{-6}$	—
$l_{14}^{(r)}$	$7 \cdot 740 \times 10^{-6}$	$1 \cdot 378 \times 10^{-6}$	—
$m_0^{(r)}$	$6 \cdot 243 \times 10^{-3}$	$-1 \cdot 088 \times 10^{-3}$	—
$m_2^{(r)}$	$-9 \cdot 153 \times 10^{-3}$	$-1 \cdot 792 \times 10^{-3}$	—
$m_4^{(r)}$	$-4 \cdot 785 \times 10^{-3}$	$-8 \cdot 180 \times 10^{-4}$	—
$m_6^{(r)}$	$-1 \cdot 801 \times 10^{-3}$	$-2 \cdot 955 \times 10^{-4}$	—
$m_8^{(r)}$	$-5 \cdot 918 \times 10^{-4}$	$-9 \cdot 599 \times 10^{-5}$	—
$m_{10}^{(r)}$	$-1 \cdot 809 \times 10^{-4}$	$-2 \cdot 948 \times 10^{-5}$	—
$m_{12}^{(r)}$	$-5 \cdot 300 \times 10^{-5}$	$-8 \cdot 766 \times 10^{-6}$	—

TABLE VIII.—Coefficients for $\lambda = 0 \cdot 4$.

	$r = 0$.	$r = 1$.	$r = 2$.	$r = 3$.
$d_0^{(r)}$	8×10^{-2}	$4 \cdot 108 \times 10^{-3}$	$-8 \cdot 19 \times 10^{-4}$	$-4 \cdot 43 \times 10^{-4}$
$d_2^{(r)}$	$6 \cdot 4 \times 10^{-3}$	$1 \cdot 419 \times 10^{-3}$	$3 \cdot 64 \times 10^{-4}$	$9 \cdot 94 \times 10^{-5}$
$d_4^{(r)}$	—	$2 \cdot 470 \times 10^{-5}$	$6 \cdot 70 \times 10^{-6}$	$1 \cdot 83 \times 10^{-6}$
$d_6^{(r)}$	—	$2 \cdot 523 \times 10^{-7}$	$7 \cdot 09 \times 10^{-8}$	$1 \cdot 91 \times 10^{-8}$
$d_8^{(r)}$	—	$2 \cdot 030 \times 10^{-9}$	$6 \cdot 19 \times 10^{-10}$	$1 \cdot 66 \times 10^{-10}$
$e_2^{(r)}$	-8×10^{-2}	$-1 \cdot 811 \times 10^{-2}$	$-4 \cdot 68 \times 10^{-3}$	$-1 \cdot 28 \times 10^{-3}$
$e_4^{(r)}$	—	$-2 \cdot 075 \times 10^{-4}$	$-5 \cdot 63 \times 10^{-5}$	$-1 \cdot 54 \times 10^{-5}$
$e_6^{(r)}$	—	$-1 \cdot 902 \times 10^{-6}$	$-5 \cdot 34 \times 10^{-7}$	$-1 \cdot 44 \times 10^{-7}$
$e_8^{(r)}$	—	$-1 \cdot 456 \times 10^{-8}$	$-4 \cdot 44 \times 10^{-9}$	$-1 \cdot 19 \times 10^{-9}$
$l_2^{(r)}$	$6 \cdot 011 \times 10^{-2}$	$1 \cdot 582 \times 10^{-2}$	$4 \cdot 35 \times 10^{-3}$	—
$l_4^{(r)}$	$1 \cdot 419 \times 10^{-2}$	$3 \cdot 893 \times 10^{-3}$	$1 \cdot 07 \times 10^{-3}$	—
$l_6^{(r)}$	$3 \cdot 548 \times 10^{-3}$	$1 \cdot 005 \times 10^{-3}$	$2 \cdot 72 \times 10^{-4}$	—
$l_8^{(r)}$	$8 \cdot 404 \times 10^{-4}$	$2 \cdot 564 \times 10^{-4}$	$6 \cdot 90 \times 10^{-5}$	—
$l_{10}^{(r)}$	$1 \cdot 951 \times 10^{-4}$	$6 \cdot 631 \times 10^{-5}$	$1 \cdot 78 \times 10^{-5}$	—
$l_{12}^{(r)}$	$4 \cdot 442 \times 10^{-5}$	$1 \cdot 734 \times 10^{-5}$	$4 \cdot 66 \times 10^{-6}$	—
$l_{14}^{(r)}$	$9 \cdot 913 \times 10^{-6}$	$4 \cdot 603 \times 10^{-6}$	$1 \cdot 24 \times 10^{-6}$	—
$m_0^{(r)}$	$1 \cdot 284 \times 10^{-2}$	$-2 \cdot 558 \times 10^{-3}$	$-1 \cdot 386 \times 10^{-3}$	—
$m_2^{(r)}$	$-1 \cdot 465 \times 10^{-2}$	$-4 \cdot 949 \times 10^{-3}$	$-1 \cdot 477 \times 10^{-3}$	—
$m_4^{(r)}$	$-7 \cdot 613 \times 10^{-3}$	$-2 \cdot 285 \times 10^{-3}$	$-6 \cdot 48 \times 10^{-4}$	—
$m_6^{(r)}$	$-2 \cdot 812 \times 10^{-3}$	$-8 \cdot 351 \times 10^{-4}$	$-2 \cdot 32 \times 10^{-4}$	—
$m_8^{(r)}$	$-9 \cdot 020 \times 10^{-4}$	$-2 \cdot 757 \times 10^{-4}$	$-7 \cdot 56 \times 10^{-5}$	—
$m_{10}^{(r)}$	$-2 \cdot 683 \times 10^{-4}$	$-8 \cdot 65 \times 10^{-5}$	$-2 \cdot 36 \times 10^{-5}$	—
$m_{12}^{(r)}$	$-7 \cdot 63 \times 10^{-5}$	$-2 \cdot 64 \times 10^{-5}$	$-7 \cdot 16 \times 10^{-6}$	—

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TABLE IX.—Coefficients for $\lambda = 0 \cdot 5$.

	$r = 0.$	$r = 1.$	$r = 2.$	$r = 3.$	$r = 4.$
$d_0^{(r)}$	$1 \cdot 25 \times 10^{-1}$	$1 \cdot 177 \times 10^{-2}$	$-1 \cdot 890 \times 10^{-3}$	$-1 \cdot 920 \times 10^{-3}$	$-9 \cdot 84 \times 10^{-4}$
$d_2^{(r)}$	$1 \cdot 5625 \times 10^{-2}$	$4 \cdot 906 \times 10^{-3}$	$1 \cdot 802 \times 10^{-3}$	$7 \cdot 07 \times 10^{-4}$	$2 \cdot 85 \times 10^{-4}$
$d_4^{(r)}$	—	$1 \cdot 901 \times 10^{-4}$	$7 \cdot 910 \times 10^{-5}$	$3 \cdot 16 \times 10^{-5}$	$1 \cdot 28 \times 10^{-5}$
$d_6^{(r)}$	—	$4 \cdot 315 \times 10^{-6}$	$2 \cdot 053 \times 10^{-6}$	$8 \cdot 17 \times 10^{-7}$	$3 \cdot 29 \times 10^{-7}$
$d_8^{(r)}$	—	$7 \cdot 46 \times 10^{-8}$	$4 \cdot 47 \times 10^{-8}$	$1 \cdot 80 \times 10^{-8}$	$7 \cdot 2 \times 10^{-9}$
$e_2^{(r)}$	$-1 \cdot 25 \times 10^{-1}$	$-4 \cdot 047 \times 10^{-2}$	$-1 \cdot 505 \times 10^{-2}$	$-5 \cdot 93 \times 10^{-3}$	$-2 \cdot 40 \times 10^{-3}$
$e_4^{(r)}$	—	$-1 \cdot 027 \times 10^{-3}$	$-4 \cdot 281 \times 10^{-4}$	$-1 \cdot 71 \times 10^{-4}$	$-6 \cdot 92 \times 10^{-5}$
$e_6^{(r)}$	—	$-2 \cdot 089 \times 10^{-5}$	$-9 \cdot 94 \times 10^{-6}$	$-3 \cdot 96 \times 10^{-6}$	$-1 \cdot 59 \times 10^{-6}$
$e_8^{(r)}$	—	$-3 \cdot 436 \times 10^{-7}$	$-2 \cdot 058 \times 10^{-7}$	$-8 \cdot 26 \times 10^{-8}$	$-3 \cdot 31 \times 10^{-8}$
$l_2^{(r)}$	$8 \cdot 831 \times 10^{-2}$	$3 \cdot 392 \times 10^{-2}$	$1 \cdot 355 \times 10^{-2}$	$5 \cdot 510 \times 10^{-3}$	—
$l_4^{(r)}$	$1 \cdot 959 \times 10^{-2}$	$8 \cdot 356 \times 10^{-3}$	$3 \cdot 364 \times 10^{-3}$	$1 \cdot 367 \times 10^{-3}$	—
$l_6^{(r)}$	$4 \cdot 618 \times 10^{-3}$	$2 \cdot 224 \times 10^{-3}$	$8 \cdot 921 \times 10^{-4}$	$3 \cdot 601 \times 10^{-4}$	—
$l_8^{(r)}$	$1 \cdot 015 \times 10^{-3}$	$5 \cdot 950 \times 10^{-4}$	$2 \cdot 403 \times 10^{-4}$	$9 \cdot 66 \times 10^{-5}$	—
$l_{10}^{(r)}$	$2 \cdot 140 \times 10^{-4}$	$1 \cdot 632 \times 10^{-4}$	$6 \cdot 67 \times 10^{-5}$	$2 \cdot 68 \times 10^{-5}$	—
$l_{12}^{(r)}$	$4 \cdot 25 \times 10^{-5}$	$4 \cdot 54 \times 10^{-5}$	$1 \cdot 89 \times 10^{-5}$	$7 \cdot 56 \times 10^{-6}$	—
$l_{14}^{(r)}$	$7 \cdot 76 \times 10^{-6}$	$1 \cdot 29 \times 10^{-5}$	$5 \cdot 48 \times 10^{-6}$	$2 \cdot 18 \times 10^{-6}$	—
$m_0^{(r)}$	$2 \cdot 355 \times 10^{-2}$	$-3 \cdot 779 \times 10^{-3}$	$-3 \cdot 841 \times 10^{-3}$	$-1 \cdot 969 \times 10^{-3}$	—
$m_2^{(r)}$	$-1 \cdot 964 \times 10^{-2}$	$-1 \cdot 017 \times 10^{-2}$	$-4 \cdot 478 \times 10^{-3}$	$-1 \cdot 891 \times 10^{-3}$	—
$m_4^{(r)}$	$-1 \cdot 010 \times 10^{-2}$	$-4 \cdot 819 \times 10^{-3}$	$-2 \cdot 012 \times 10^{-3}$	$-8 \cdot 30 \times 10^{-4}$	—
$m_5^{(r)}$	$-3 \cdot 610 \times 10^{-3}$	$-1 \cdot 809 \times 10^{-3}$	$-7 \cdot 425 \times 10^{-4}$	$-3 \cdot 03 \times 10^{-4}$	—
$m_8^{(r)}$	$-1 \cdot 108 \times 10^{-3}$	$-6 \cdot 170 \times 10^{-4}$	$-2 \cdot 524 \times 10^{-4}$	$-1 \cdot 02 \times 10^{-4}$	—
$m_{10}^{(r)}$	$-3 \cdot 122 \times 10^{-4}$	$-2 \cdot 014 \times 10^{-4}$	$-8 \cdot 26 \times 10^{-5}$	$-3 \cdot 34 \times 10^{-5}$	—
$m_{12}^{(r)}$	$-8 \cdot 31 \times 10^{-5}$	$-6 \cdot 41 \times 10^{-5}$	$-2 \cdot 65 \times 10^{-5}$	$-1 \cdot 07 \times 10^{-5}$	—

The approximation is in each case carried out until the greatest residual stress on the edge of the strip is 1 per cent. of T or less. With $\lambda = 0 \cdot 5$ the inclusion of χ_8 is barely sufficient, the residual stress $\widehat{yy}$ at the point on the edge where it is met by the y-axis being about $1 \cdot 1$ per cent. of T. Since the distribution of stress near the hole is the matter of greatest interest, the approximation is always ended with a stress function of even order so that the hole is free from residual tractions.

The various stress functions have now to be summed to give the final value of χ. This may be written—

$$\chi = \tfrac{1}{4} b^2 \mathrm{T} \rho^2 (1 + \cos 2\theta)$$
$$+ b^2 \mathrm{T} \left[- d_0 \log \rho + m_0 \rho^2 + \sum_{n=1}^{\infty} \left\{ \frac{d_{2n}}{\rho^2} + \frac{e_{2n}}{\rho^{2n-2}} + (l_{2n} + m_{2n} \rho^2) \rho^{2n} \right\} \cos 2n\theta \right], \quad . \quad (59)$$

where

$$\left. \begin{aligned} d_0 &= \sum_r d_0^{(r)}, & d_{2n} &= \sum_r d_{2n}^{(r)} \\ e_{2n} &= \sum_r e_{2n}^{(r)}, & l_{2n} &= \sum_r l_{2n}^{(r)} \\ m_0 &= \sum_r m_0^{(r)}, & m_{2n} &= \sum_r m_{2n}^{(r)} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

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The stresses are

$$\left.\begin{aligned}
\widehat{rr} &= \mathrm{T}\Bigg[\tfrac{1}{2}(1-\cos 2\theta) + 2m_0 - \frac{d_0}{\rho^2} \\
&\quad - 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)\,d_{2n}}{\rho^{2n+2}} + \frac{(n+1)(2n-1)\,e_{2n}}{\rho^{2n}}\right. \\
&\qquad \left. + n(2n-1)\,l_{2n}\rho^{2n-2} + (n-1)(2n+1)\,m_{2n}\rho^{2n}\right\}\cos 2n\theta\Bigg] \\
\widehat{\theta\theta} &= \mathrm{T}\Bigg[\tfrac{1}{2}(1+\cos 2\theta) + 2m_0 + \frac{d_0}{\rho^2} \\
&\quad + 2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)\,d_{2n}}{\rho^{2n+2}} + \frac{(n-1)(2n-1)\,e_{2n}}{\rho^{2n}}\right. \\
&\qquad \left. + n(2n-1)\,l_{2n}\rho^{2n-2} + (n+1)(2n+1)\,m_{2n}\rho^{2n}\right\}\cos 2n\theta\Bigg] \\
\widehat{r\theta} &= \mathrm{T}\Bigg[\tfrac{1}{2}\sin 2\theta + 2\sum_{n=1}^{\infty}\left\{n(2n-1)\left(l_{2n}\rho^{2n-2} - \frac{e_{2n}}{\rho^{2n}}\right)\right. \\
&\qquad \left. + n(2n+1)\left(m_{2n}\rho^{2n} - \frac{d_{2n}}{\rho^{2n+2}}\right)\right\}\sin 2n\theta\Bigg]
\end{aligned}\right\} \quad . \quad . \quad (61)$$

These equations, together with Table X, which gives the coefficients for various values of λ, give the complete solution for the stresses in the neighbourhood of the hole.

TABLE X.—Coefficients in the Final Value of the Stress Function.*

	$\lambda = 0{\cdot}1$.	$\lambda = 0{\cdot}2$.	$\lambda = 0{\cdot}3$.	$\lambda = 0{\cdot}4$.	$\lambda = 0{\cdot}5$.
d_0	$5{\cdot}01 \times 10^{-3}$	$2{\cdot}02 \times 10^{-2}$	$4{\cdot}59 \times 10^{-2}$	$8{\cdot}28 \times 10^{-2}$	$1{\cdot}32 \times 10^{-1}$
d_2	$2{\cdot}54 \times 10^{-5}$	$4{\cdot}25 \times 10^{-4}$	$2{\cdot}34 \times 10^{-3}$	$8{\cdot}28 \times 10^{-3}$	$2{\cdot}33 \times 10^{-2}$
d_4	$3{\cdot}15 \times 10^{-11}$	$3{\cdot}05 \times 10^{-8}$	$1{\cdot}85 \times 10^{-6}$	$3{\cdot}32 \times 10^{-5}$	$3{\cdot}14 \times 10^{-4}$
d_6	$1{\cdot}40 \times 10^{-15}$	$2{\cdot}13 \times 10^{-11}$	$6{\cdot}32 \times 10^{-9}$	$3{\cdot}42 \times 10^{-7}$	$7{\cdot}51 \times 10^{-6}$
d_8	$5{\cdot}00 \times 10^{-20}$	$1{\cdot}19 \times 10^{-14}$	$1{\cdot}72 \times 10^{-11}$	$2{\cdot}82 \times 10^{-9}$	$1{\cdot}45 \times 10^{-7}$
e_2	$-5{\cdot}08 \times 10^{-3}$	$-2{\cdot}13 \times 10^{-2}$	$-5{\cdot}21 \times 10^{-2}$	$-1{\cdot}04 \times 10^{-1}$	$-1{\cdot}89 \times 10^{-1}$
e_4	$-4{\cdot}21 \times 10^{-9}$	$-1{\cdot}02 \times 10^{-6}$	$-2{\cdot}75 \times 10^{-5}$	$-2{\cdot}79 \times 10^{-4}$	$-1{\cdot}70 \times 10^{-3}$
e_6	$-1{\cdot}68 \times 10^{-13}$	$-6{\cdot}41 \times 10^{-10}$	$-8{\cdot}45 \times 10^{-8}$	$-2{\cdot}58 \times 10^{-6}$	$-3{\cdot}64 \times 10^{-5}$
e_8	$-5{\cdot}72 \times 10^{-18}$	$-3{\cdot}41 \times 10^{-13}$	$-2{\cdot}20 \times 10^{-10}$	$-2{\cdot}02 \times 10^{-8}$	$-6{\cdot}7 \times 10^{-7}$
l_2	$4{\cdot}13 \times 10^{-3}$	$1{\cdot}62 \times 10^{-2}$	$4{\cdot}09 \times 10^{-2}$	$8{\cdot}03 \times 10^{-2}$	$1{\cdot}40 \times 10^{-1}$
l_4	$1{\cdot}06 \times 10^{-3}$	$4{\cdot}10 \times 10^{-3}$	$1{\cdot}01 \times 10^{-2}$	$1{\cdot}92 \times 10^{-2}$	$3{\cdot}27 \times 10^{-2}$
l_6	$2{\cdot}83 \times 10^{-4}$	$1{\cdot}08 \times 10^{-3}$	$2{\cdot}61 \times 10^{-3}$	$4{\cdot}82 \times 10^{-3}$	$8{\cdot}09 \times 10^{-3}$
l_8	$7{\cdot}24 \times 10^{-5}$	$2{\cdot}74 \times 10^{-4}$	$6{\cdot}44 \times 10^{-4}$	$1{\cdot}17 \times 10^{-3}$	$1{\cdot}95 \times 10^{-3}$
l_{10}	$1{\cdot}82 \times 10^{-5}$	$6{\cdot}82 \times 10^{-5}$	$1{\cdot}57 \times 10^{-4}$	$2{\cdot}79 \times 10^{-4}$	$4{\cdot}71 \times 10^{-4}$
l_{12}	$4{\cdot}65 \times 10^{-6}$	$1{\cdot}80 \times 10^{-5}$	$3{\cdot}80 \times 10^{-5}$	$6{\cdot}64 \times 10^{-5}$	$1{\cdot}14 \times 10^{-4}$
l_{14}	$1{\cdot}15 \times 10^{-6}$	$4{\cdot}41 \times 10^{-6}$	$9{\cdot}12 \times 10^{-6}$	$1{\cdot}58 \times 10^{-6}$	$2{\cdot}8 \times 10^{-5}$
m_0	$5{\cdot}69 \times 10^{-4}$	$2{\cdot}46 \times 10^{-3}$	$5{\cdot}15 \times 10^{-3}$	$8{\cdot}89 \times 10^{-3}$	$1{\cdot}40 \times 10^{-2}$
m_2	$-1{\cdot}13 \times 10^{-3}$	$-4{\cdot}35 \times 10^{-3}$	$-1{\cdot}09 \times 10^{-2}$	$-2{\cdot}11 \times 10^{-2}$	$-3{\cdot}62 \times 10^{-2}$
m_4	$-5{\cdot}95 \times 10^{-4}$	$-2{\cdot}29 \times 10^{-3}$	$-5{\cdot}60 \times 10^{-3}$	$-1{\cdot}06 \times 10^{-2}$	$-1{\cdot}78 \times 10^{-2}$
m_6	$-2{\cdot}28 \times 10^{-4}$	$-8{\cdot}70 \times 10^{-4}$	$-2{\cdot}10 \times 10^{-3}$	$-3{\cdot}88 \times 10^{-3}$	$-6{\cdot}47 \times 10^{-3}$
m_8	$-7{\cdot}65 \times 10^{-5}$	$-2{\cdot}90 \times 10^{-4}$	$-6{\cdot}88 \times 10^{-4}$	$-1{\cdot}25 \times 10^{-3}$	$-2{\cdot}08 \times 10^{-3}$
m_{10}	$-2{\cdot}39 \times 10^{-5}$	$-8{\cdot}99 \times 10^{-5}$	$-2{\cdot}10 \times 10^{-4}$	$-3{\cdot}78 \times 10^{-4}$	$-6{\cdot}30 \times 10^{-4}$
m_{12}	$-7{\cdot}30 \times 10^{-6}$	$-2{\cdot}86 \times 10^{-5}$	$-6{\cdot}17 \times 10^{-5}$	$-1{\cdot}10 \times 10^{-4}$	$-1{\cdot}84 \times 10^{-4}$

* Most of the entries in this table would be modified if the series for χ were carried further, but the effects on the calculated stresses would be slight.

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The Numerical Values of the Stresses.

The problem of greatest interest is that of finding the stresses at the circumference of the hole. Here we have $\widehat{rr} = \widehat{r\theta} = 0$ while, putting $\rho = \lambda$ in the second of equations (61),

$$\widehat{\theta\theta} = \mathrm{T}\,[p_0 + p_2 \cos 2\theta + p_4 \cos 4\theta + \ldots], \quad \ldots \quad (62)$$

where

$$\left.\begin{aligned} p_0 &= \tfrac{1}{2} + 2m_0 + d_0/\lambda^2 \\ p_2 &= \tfrac{1}{2} + 6d_2/\lambda^4 + 2l_2 + 12m_2\lambda^2 \\ p_4 &= 20d_4/\lambda^6 + 6e_4/\lambda^4 + 12l_4\lambda^2 + 30m_4\lambda^4 \\ p_6 &= 42d_6/\lambda^8 + 20e_6/\lambda^6 + 30l_6\lambda^4 + 56m_6\lambda^6 \\ p_8 &= 72d_8/\lambda^{10} + 42e_8/\lambda^8 + 56l_8\lambda^6 + 90m_8\lambda^8 \\ &\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots \quad (63)$$

The values of these coefficients are shown in Table XI and the consequent values of $\widehat{\theta\theta}$ for values of θ from 0° to 90° at intervals of 15° are given in Table XII. It will be seen that the stress at $\theta = 0$ is, when $\lambda = 0\cdot5$, about $4\frac{1}{3}$ times T as compared with 3T for a hole which is very small compared with the width of the plate, while the negative value of $\widehat{\theta\theta}$ at $\theta = 90°$ is over 1·5T as compared with T for a very small hole.

TABLE XI.—Coefficients in equation for $\widehat{\theta\theta}$ at rim of hole.

	$\lambda = 0\cdot1$.	$\lambda = 0\cdot2$.	$\lambda = 0\cdot3$.	$\lambda = 0\cdot4$.	$\lambda = 0\cdot5$.
p_0	1·00	1·01	1·02	1·04	1·06
p_2	2·03	2·12	2·30	2·56	2·91
p_4	—	0·01	0·04	0·13	0·30
p_6	—	—	—	0·01	0·04
p_8	—	—	—	—	0·01

TABLE XII (see fig. 2, p. 83).—Values of $\dfrac{\widehat{\theta\theta}}{\mathrm{T}}$ at rim of hole.*

θ.	$\lambda = 0$.	$\lambda = 0\cdot1$.	$\lambda = 0\cdot2$.	$\lambda = 0\cdot3$.	$\lambda = 0\cdot4$.	$\lambda = 0\cdot5$.
0°	3·00	3·03	3·14	3·36	3·74	4·32
15°	2·73	2·74	2·85	3·03	3·32	3·72
30°	2·00	2·01	2·07	2·15	2·25	2·32
45°	1·00	1·00	1·00	0·98	0·91	0·77
60°	0·00	— 0·01	— 0·06	— 0·15	— 0·30	— 0·51
75°	— 0·73	— 0·74	— 0·82	— 0·95	— 1·12	— 1·32
90°	— 1·00	— 1·03	— 1·11	— 1·26	— 1·44	— 1·58

* In this and in later Tables of stresses the second decimal place may at times be inaccurate.

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These values are considerably higher than those found by COKER, CHAKKO and SATAKE* by the methods of photo-elasticity. Applying a tension of 556 lbs./in.² to a strip of breadth 1 inch containing a hole of diameter $\frac{1}{2}''$, they found at the rim of the hole the values of $\widehat{\theta\theta}$ shown in the second column of Table XIII. The ratios of these to the applied tension are given in the third column and the values calculated by the present method are repeated in the fourth column. The discrepancy is probably due to the very rapid variation of the stress near the hole. If the value of $\widehat{\theta\theta}$ on a circle of radius $\rho = 0 \cdot 55''$ is calculated it is found to be given by equation (62), with the following values for the coefficients :

$$p_0 = 0 \cdot 97, \qquad p_2 = 1 \cdot 93, \qquad p_4 = 0 \cdot 18, \qquad p_6 = 0 \cdot 02, \qquad p_8 = 0 \cdot 00,$$

and these lead to the values of $\widehat{\theta\theta}$ in the last column of Table XIII. These are much lower than the stresses found at the rim of the hole by the optical method. It is clear, therefore, that if the measurements made corresponded to points even at a very short distance from the hole the estimates of $\widehat{\theta\theta}$ would be considerably too low.

TABLE XIII (*see* fig. 3, p. 83).—Comparison of the above results with those obtained by COKER, CHAKKO and SATAKE, using optical methods.

θ.	Stress at rim of hole ($\rho = \lambda = 0 \cdot 5$) found by optical method.		Stress at rim of hole and on a neighbouring circle ($\rho = 0 \cdot 55$) by calculation.	
	$\widehat{\theta\theta}$.	$\widehat{\theta\theta}$/T.	$\widehat{\theta\theta}$/T for $\rho = 0 \cdot 5$.	$\widehat{\theta\theta}$/T for $\rho = 0 \cdot 55$.
0	1960	3·53	4·32	3·34
15°	1800	3·24	3·72	2·94
30°	1190	2·12	2·32	1·94
45°	490	0·88	0·77	0·78
60°	— 280	— 0·50	— 0·51	— 0·22
75°	— 600	— 1·08	— 1·32	— 0·84
90°	— 700	— 1·26	— 1·58	— 1·06

It is also of great interest to determine the distribution of tension across the minimum section. This is found by putting $\theta = 0$ in the equation for $\widehat{\theta\theta}$, the result being

$$\begin{aligned}\widehat{\theta\theta} = \mathrm{T}\Big[&(1 + 2m_0 + 2l_2) + 12\,(m_2 + l_4)\,\rho^2 + 30\,(m_4 + l_6)\,\rho^4 \\ &+ 56\,(m_6 + l_8)\,\rho^6 + 90\,(m_8 + l_{10})\,\rho^8 + \dots \\ &+ \frac{d_0}{\rho^2} + \frac{6\,(d_2 + e_4)}{\rho^4} + \frac{20\,(d_4 + e_6)}{\rho^6} \\ &+ \frac{42\,(d_6 + e_8)}{\rho^8} + \dots\Big]. \qquad (64)\end{aligned}$$

* "Photo-Elastic and Strain Measurements of the Effects of Circular Holes on the Distribution of Stress in Tension Members," 'Institution of Engineers and Shipbuilders in Scotland,' 1920.

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When the values of the coefficients in Table X are inserted into (64) the limited nature of the convergence becomes apparent. In order that values of $\widehat{\theta\theta}$ on the edge of the strip, *i.e.*, for $\rho = 1$, may be found correctly to the second decimal place, it is necessary to include positive powers of ρ beyond the 12th, and this requires more coefficients than have been given in Table X, but the error made in stopping at ρ^{12} will be small.

The values obtained are shown in Table XIV. A remarkable feature is the very rapid drop in the value of the tension near the edge of the strip. This drop is greater than was revealed by optical methods. With a hole of diameter equal to half that of the strip, Coker, Chakko and Satake found a tension slightly greater than T on the edge of the strip at the minimum section, so that their measurements again seem to correspond to a distance of about 0·05″ from the boundary.

Table XIV (*see* fig. 4, p. 83).—Values of $\widehat{\theta\theta}/T$ at points on the minimum section.

ρ.	λ = 0·1.	λ = 0·2.	λ = 0·3.	λ = 0·4.	λ = 0·5.
0·1	3·03				
0·2	1·23	3·14			
0·3	1·08	1·57	3·36		
0·4	1·04	1·26	1·93	3·74	
0·5	1·03	1·16	1·47	2·30	4·32
0·6	1·02	1·11	1·28	1·75	2·75
0·7	1·01	1·07	1·17	1·48	2·04
0·8	1·01	1·05	1·07	1·28	1·61
0·9	1·00	1·01	0·96	1·08	1·22
1·0	0·99	0·97	0·89	0·81	0·73

The difficulty of making calculations outside the circle $\rho=1$ is a more serious hindrance when stresses in directions other than $\theta = 0$ are considered. For example, the tension on the axis is given by putting $\theta = \pi/2$ in the formula for $\widehat{rr}$ in (61): this leads to the series

$$\widehat{rr} = T\Big[(1 + 2l_2 + 2m_0) - 12l_4\rho^2 + 10\,(3l_6 - m_4)\,\rho^4 - 28\,(2l_8 - m_6)\,\rho^6 + 18\,(5l_{10} - 3m_8)\,\rho^8 - 44\,(3l_{12} - 2m_{10})\,\rho^{10} + 26\,(7l_{14} - 5\,m_{12})\,\rho^{12} - \ldots - \frac{d_0 - 4e_2}{\rho^2} + \frac{6\,(d_2 - 3e_4)}{\rho^4} - \frac{20\,(d_4 - 2e_6)}{\rho^6} + \frac{14\,(3d_6 - 5e_8)}{\rho^8} - \ldots\Big] \quad \ldots \ldots \quad (65)$$

With all the terms given in (65), and no more can be included without extending the list of coefficients in Table X, it is not possible to advance far beyond $\rho = 1$. The values of the tension for $\lambda = 0\cdot5$ are given in the first column of Table XV. In the second column are shown the corresponding values for a plate of infinite width. In

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order that the comparison may be carried beyond $\rho = 1$, some values of the tension in the strip have been calculated by another method. We first write the value of the stress function in the form

$$\chi = \bar{\chi}_{2r} + \bar{\chi}_{2r+1} \quad \ldots \ldots \ldots \quad (66)$$

where

$$\bar{\chi}_{2r} = \chi_0' + \chi_0 + \chi_2 + \chi_4 + \quad \ldots \ldots \ldots \quad (67)$$

and will correspond to a constant term and negative powers in the series for the stresses, while

$$\bar{\chi}_{2r+1} = \chi_1 + \chi_3 + \chi_5 + \ldots \quad \ldots \ldots \ldots \quad (68)$$

and corresponds to a constant term and positive powers. When this part of the series is unsuitable for calculation, the terms on the right of (68) must be regarded as defined by the integral formula (27). Adding these terms and using equations (24) to (26) and (58) and (60), we have

$$\bar{\chi}_{2r+1} = 4b^2\mathrm{T}\left[\int_0^\infty \frac{\eta \mathrm{S} \cos u\xi}{\Sigma}(s\bar{\Phi} + c\bar{\Psi})\,du \right.$$
$$\left. - 4\int_0^\infty \frac{\mathrm{C}\cos u\xi}{\Sigma}\left(\frac{s+uc}{u}\bar{\Phi} + s\bar{\Psi}\right)du\right] \quad \ldots \ldots \quad (69)$$

where

$$\bar{\Phi} = e^{-u}\left[\tfrac{1}{2}d_0 + \tfrac{1}{2}\sum_{n=1}^{\infty}(d_{2n} - e_{2n+2})\frac{u^{2n}}{(2n-1)!} + \sum_{n=1}^{\infty} e_{2n}\frac{u^{2n-1}}{(2n-2)!}\right] \ldots \quad (70)$$

$$\bar{\Psi} = e^{-u}\left[\frac{d_0 - 2e_2}{2} + \sum_{n=1}^{\infty}\frac{nd_{2n} - (n+1)\,e_{2n+2}}{(2n)!}\,.\,u^{2n} + \sum_{n=1}^{\infty}\frac{e_{2n}}{(2n-2)!}u^{2n-1}\right] \quad (71)$$

Expansions of (69) about other points than the origin may now be obtained. Unfortunately, the coefficients in these expansions cannot be wholly expressed in terms of the integrals so far tabulated and the importance of a detailed discussion of the stresses outside the circle $\rho = 1$ does not seem to be great enough to warrant a fresh tabulation of integrals. An alternative is to determine the stresses at selected points by a direct computation of the integrals obtained by differentiating (69). The additional entries in Table XV were obtained in this way.

The stress in the direction of the x-axis is given by

$$\widehat{xx} = \frac{1}{b^2}\left(\frac{\partial^2 \bar{\chi}_{2r}}{\partial \eta^2} + \frac{\partial^2 \bar{\chi}_{2r+1}}{\partial \eta^2}\right)$$
$$= \widehat{xx}_1 + 4\mathrm{T}\left[\int_0^\infty \frac{u^2\eta\mathrm{S} + 2u\mathrm{C}}{\Sigma}\cos u\xi\,.\,(s\bar{\Phi} + c\bar{\Psi})\,du \right.$$
$$\left. - \int_0^\infty \frac{u^2\mathrm{C}\,.\cos u\xi}{\Sigma}\left(\frac{s+uc}{u}\bar{\Phi} + s\bar{\Psi}\right)du\right], \quad \ldots \quad (72)$$

where $\widehat{xx}_1$ is easily derived from the appropriate part of (59).

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On the line $\eta = 0$, this reduces to

$$\widehat{xx} = \widehat{rr} = \mathrm{T}\left[1 - \frac{d_0 - 4e_2}{\rho^2} + \ldots + 4\mathrm{P}\right] \quad \ldots \ldots \ldots \quad (73)$$

the negative powers having the same coefficients as in (65) while

$$\mathrm{P} = \int_0^\infty \{(s - uc)\,\overline{\Phi} + (2c - us)\,\overline{\Psi}\}\,\frac{u\,.\cos u\xi}{\Sigma}\,du. \quad \ldots \ldots \quad (74)$$

Integrals of this kind are readily computed by the use of FILON's extension of SIMPSON's formula.* The integral in (74) is rapidly convergent, the effective upper limit being 5 or 6. Since the values of $\overline{\Phi}$ and $\overline{\Psi}$ can be found only to 3 significant figures, the value of P is unlikely to be correct to more than 2 figures, but this does not matter as, when ρ exceeds 1, the contribution from P to $\widehat{xx}$ is a small one. The values obtained were—

ξ	P
$\pi/2$	0·037
$2\pi/3$	0·035
π	0·022

They lead to the last three entries in the first column of Table XV. This table reveals the interesting fact that the tension on the axis tends to its limiting value more rapidly than in a plate of great breadth.

TABLE XV.—Stresses on the axis of a strip whose diameter is twice that of the hole compared with those in very broad plate with the same size of hole.

ξ	$\widehat{xx}/\mathrm{T}$ $(= \widehat{rr}/\mathrm{T})$.		$\widehat{yy}/\mathrm{T}$ $(= \widehat{\theta\theta}/\mathrm{T})$.	
	Strip.	Infinite Plate.	Strip.	Infinite Plate.
0·5	0·00	0·00	− 1·6	− 1·00
0·6	−0·09	−0·01	− 0·7	− 0·38
0·7	0·02	0·12	− 0·4	− 0·14
0·8	0·17	0·25	− 0·1	− 0·03
0·9	0·31	0·37	0·0	0·01
1·0	0·44	0·47	+ 0·1	0·03
$\pi/2$	0·81	0·76	—	—
$2\pi/3$	0·95	0·86	—	—
π	1·00	0·94	—	—

The cross-stress on the axis is given by putting $\theta = \pi/2$ in the formula for $\widehat{\theta\theta}$ in (61). The calculated values, to the first decimal place, are shown in the third column of

* L. N. G. FILON, 'Proc. Roy. Soc., Edin.,' vol. 49, pp. 38–47 (1929).

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Table XV, and those for an infinite plate with a hole of the same size are given in the last column. It does not seem necessary to trace these stresses further. In the infinite plate, after $\rho = 1$ is passed, $\widehat{yy}$ increases very slightly to a maximum and then decreases towards zero. There is no reason to doubt that the value of $\widehat{yy}$ in the strip will behave in a similar way.

It is of much greater interest to determine the values of the tension in the edge of the strip and to see in what way it passes from the very small value near the hole to its asymptotic value at infinity. Coker, Chakko and Satake* found by optical methods that the value of $\widehat{xx}$ at $\xi = 0$ is a minimum, and that a large maximum occurs at a point further along the edge.

In order to discover whether their results are confirmed by the present method, we return to equation (72) and put $\eta = 1$. After a little reduction, the equation becomes

$$\widehat{xx} = \widehat{xx}_1 + 4\mathrm{T}\int_0^\infty \{(cs - u)\,\overline{\Phi} + 2c^2\overline{\Psi}\}\,\frac{u\,.\,\cos u\xi}{\Sigma}\,du. \quad . \quad . \quad . \quad . \quad (75)$$

This integral does not converge as rapidly as the one in (74) and a further reduction is helpful. This is obtained by using the relations†

$$\left.\begin{aligned} sc - u &= \tfrac{1}{2}\Sigma - 2u \\ 2c^2 &= \Sigma - 2u + e^{-2u} + 1 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

and then separating out the terms from which Σ can be removed. We then have

$$\widehat{xx} = \widehat{xx}_1 + 4\mathrm{T}\int_0^\infty (\tfrac{1}{2}\overline{\Phi} + \overline{\Psi})\,u\cos u\xi du$$
$$- 4\mathrm{T}\int_0^\infty \{2u\,(\overline{\Phi} + \overline{\Psi}) - (1 + e^{-2u})\,\overline{\Psi}\}\,\frac{u\cos u\xi}{\Sigma}\,du. \quad (77)$$

The second integral is now very rapidly convergent, while the first may be evaluated in a convenient form. For, when the values of $\overline{\Phi}$ and $\overline{\Psi}$ are inserted from equations (70) and (71) the integral reduces to a series of integrals of the type

$$\int_0^\infty e^{-u}\,.\,u^n\,.\cos u\xi\,.\,du.$$

These integrals are well known, they may be evaluated by taking the elementary type

$$\int_0^\infty e^{-au}\,.\cos u\xi du = \frac{a}{a^2 + \xi^2},$$

* *Loc. cit.*

† *Cf.* p. 107 of the Author's paper cited above.

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and differentiating n times with respect to a. This leads at once to the equation

$$\int_0^\infty e^{-u} \,.\, u^n \cos u\xi \,.\, du = \frac{n\,!\,\cos(n+1)\,\theta}{(1+\xi^2)^{\frac{n+1}{2}}}$$

$$= \frac{n\,!\,\cos(n+1)\,\theta}{\rho^{n+1}}, \quad \ldots \ldots \quad (78)$$

where

$$\left.\begin{aligned} \tan\theta &= \xi \\ \rho &= \sqrt{1+\xi^2} \end{aligned}\right\}, \quad \ldots \ldots \quad (79)$$

so that $\rho \,.\, \theta$ are the polar co-ordinates of a point on the edge of the strip. Using this result we easily obtain

$$4\int_0^\infty (\tfrac{1}{2}\overline{\Phi} + \overline{\Psi})\, u \cos u\xi \,.\, du = (3d_0 - 4e_2)\frac{\cos 2\theta}{\rho^2}$$

$$+ 2\sum_{n=1}^{\infty}(2n+1)\{3nd_{2n} - (3n+2)\,e_{2n+2}\}\frac{\cos(2n+2)\,\theta}{\rho^{2n+2}}$$

$$+ 6\sum_{n=1}^{\infty} 2n\,(2n-1)\,e_{2n}\frac{\cos(2n+1)\,\theta}{\rho^{2n+1}}. \quad \ldots \ldots \quad (80)$$

The formula for $\widehat{xx}_1$ may be written down by making a few obvious changes of notation in (12) and adding the uniform tension term due to χ_0'; it is

$$\widehat{xx}_1 = \mathrm{T}\left[1 + \frac{d_0 - 2e_2}{\rho^2}\cos 2\theta + 2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)\,(nd_{2n} - e_{2n+2})}{\rho^{2n+2}}\right.\right.$$

$$\left.\left. + \frac{n\,(2n-1)\,e_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\,\theta\right]. \quad \ldots \quad (81)$$

When these results are substituted into (77) and the terms are collected, it becomes

$$\widehat{xx} = \mathrm{T}\left[+1\ \frac{2\,(2d_0 - 3e_2)}{\rho^2}\cos 2\theta\right.$$

$$+ 2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)\,(4nd_{2n} - \overline{3n+1}\ e_{2n+2})}{\rho^{2n+2}} + \frac{n\,(2n-1)\,e_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\,\theta$$

$$\left. + 6\sum_{n=1}^{\infty}\frac{2n\,(2n-1)\,e_{2n}}{\rho^{2n+1}}\cos(2n+1)\,\theta - 4\mathrm{Q}\right], \quad \ldots \quad (82)$$

where

$$\mathrm{Q} = \int_0^\infty \{2u\,(\overline{\Phi} + \overline{\Psi}) - (1 + e^{-2u})\,\Psi\}\frac{u\cos u\xi}{\Sigma} \,.\, du. \quad \ldots \quad (83)$$

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If (82) is written as

$$\widehat{xx} = \mathrm{T}\,(\mathrm{R} - 4\mathrm{Q}), \quad \ldots \ldots \ldots \ldots \quad (84)$$

and if the numerical values of the coefficients for $\lambda = 0\cdot 5$ are inserted, we find—

$$\begin{aligned} \mathrm{R} = 1 &+ \frac{1\cdot 66}{\rho^2}\cos 2\theta - \frac{2\cdot 27}{\rho^3}\cos 3\theta \\ &+ \left(\frac{0\cdot 62}{\rho^4} - \frac{0\cdot 38}{\rho^2}\right)\cos 4\theta - \frac{0\cdot 12}{\rho^5}\cos 5\theta \\ &+ \left(\frac{0\cdot 03}{\rho^6} - \frac{0\cdot 02}{\rho^4}\right)\cos 6\theta - \frac{0\cdot 01}{\rho^7}\cos 7\theta \\ &+ \ldots \quad \ldots \ldots \ldots \ldots \quad (85) \end{aligned}$$

The values of Q are most readily calculated by FILON's formula* if ξ is a simple sub-multiple of π, and as this does not much increase the difficulty of calculating R, the interval chosen for ξ was $\pi/18$, instead of a simple decimal. The values of Q, R and $\widehat{xx}$ are given in Table XVI. The expected maximum occurs and is remarkably large, for the tension rises to $1\cdot 86$ times its asymptotic value at a distance from the central section of about $0\cdot 7$ of the semi-diameter of the strip. The maximum obtained by photo-elastic methods was about $1\cdot 6$ times the applied tension and occurred rather further from the central section. The discrepancy is probably due once more to the difficulty of making optical measurements very close to the boundary.

TABLE XVI.—Values of the terms in equation (84) and of the tension in the edge of the strip. ($\lambda = 0\cdot 5$)

ξ	R.	Q.	$\widehat{xx}$/T.
0	0·51	− 0·051	0·71
$\pi/18 = 0\cdot 17$	0·73	− 0·051	0·93
$\pi/9 = 0\cdot 35$	1·19	− 0·047	1·38
$\pi/6 = 0\cdot 52$	1·57	− 0·042	1·74
$2\pi/9 = 0\cdot 70$	1·72	− 0·036	1·86
$5\pi/18 = 0\cdot 87$	1·71	− 0·029	1·83
$\pi/3 = 1\cdot 05$	1·59	− 0·024	1·69
$\pi/2 = 1\cdot 57$	1·18	− 0·016	1·24

The value of $\widehat{xx}$ at the central section is slightly lower than the value given for $\widehat{\theta\theta}$ at the same point in Table XIV. The present method of calculation is probably the more reliable. It serves to emphasise still further the rapidity with which the tension varies on the central section near the edge. This would make an optical determination of the stress extremely difficult.

* *Loc. cit. ante.*

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Summary.

The problem of determining the stresses in a perforated strip under tension is completely solved by equations (61) if only the immediate neighbourhood of the hole is considered. The series become unsuitable for purposes of calculation outside a circle concentric with the hole and of diameter equal to that of the strip. The positive powers must then be replaced by an integral formula. The coefficients in the series have to be calculated by a process of successive approximation for which general formulæ are given. These lead rapidly to the required values if the ratio (λ) of the diameter of the hole to that of the strip does not exceed 0·5. The coefficients are tabulated for values 0·1, 0·2, 0·3, 0·4, 0·5 of λ and could be estimated for intermediate values of λ by interpolation.

For the larger values of λ the concentration of stress at the boundary of the hole is very high, the greatest value when $\lambda = 0\cdot5$ being nearly $4\frac{1}{3}$ times the applied tension. A corresponding low tension, less than $\frac{3}{4}$ of that at infinity, occurs at the edge of the strip on the minimum section. At a short distance along the edge the tension rises to nearly twice its asymptotic value. The phenomena are qualitatively identical with those revealed by optical methods, but the variations in stress are more extreme. The values on the boundaries differ greatly from those at a short distance inside the material, and it seems likely that the difficulty of making optical measurements very near the boundary has led to the values of the stress at about 0·05″ from the boundary being taken as the boundary values.

The method used may be applied to more general stress-systems in a perforated strip. But the formulæ so far developed apply only to systems symmetrical about both co-ordinate axes, the edges of the strip being parallel to one axis, and the tension problem is the only one of much interest that possesses this kind of symmetry. Similar methods may lead to the solution of other problems of bi-harmonic analysis in the region between two boundaries.

In conclusion, I have to thank Prof. L. N. G. FILON for his continued interest in this work. I am indebted also to Mr. A. C. STEVENSON for verifying some important parts of the analysis.

Added 23*rd September*, 1929.—It has been suggested to me that I should add a few diagrams in illustration of my results. The stresses at the rim of the hole, recorded in Table XII, p. 74, are shown graphically in fig. 2. They are plotted on a polar diagram, tensions being denoted by radii drawn outwards from the rim of the hole and pressures by radii drawn inwards. A comparison with the stresses found experimentally by COKER, CHAKKO and SATAKE* is made in fig. 3, which corresponds to Table XIII, p. 75. In fig. 4 are shown the stresses across the minimum section for different values of λ; the corresponding numbers are to be found in Table XIV, p. 76.

* *Loc. cit. ante.*

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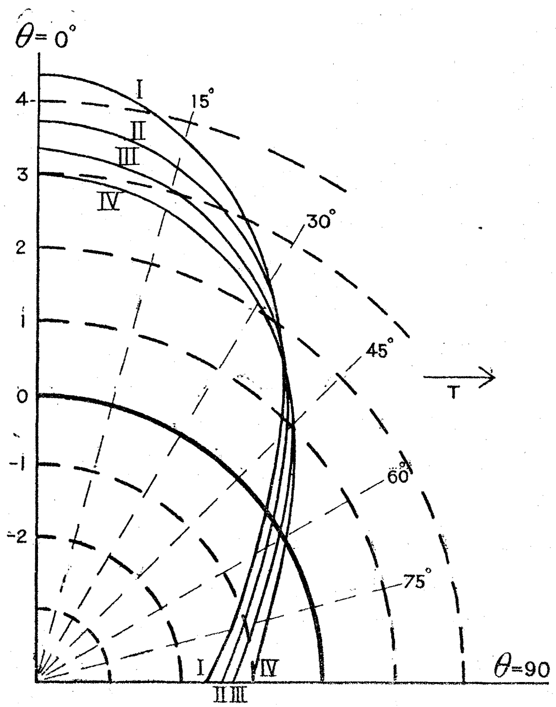

FIG. 2.—Curves showing the variations of stress at the rim of the hole for different values of λ. (Table XII, p. 74). I, $\lambda = 0{\cdot}5$; II, $\lambda = 0{\cdot}4$; III, $\lambda = 0{\cdot}3$; IV, $\lambda = 0$. Tensions are measured radially outwards from the hole; pressures radially inwards.

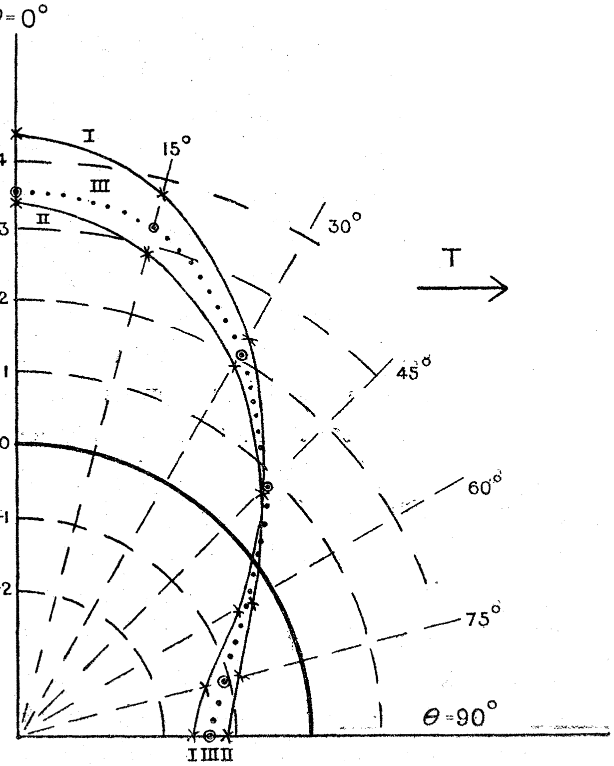

FIG. 3.—I is the same as in fig. 2 and gives the stress at the rim of the hole for $\lambda = 0{\cdot}5$. II gives the stress on a circle slightly greater than the hole ($\rho = 0{\cdot}55$). III gives the stress found at the rim of the hole by optical methods. (*cf.* Table XIII, p. 75.)

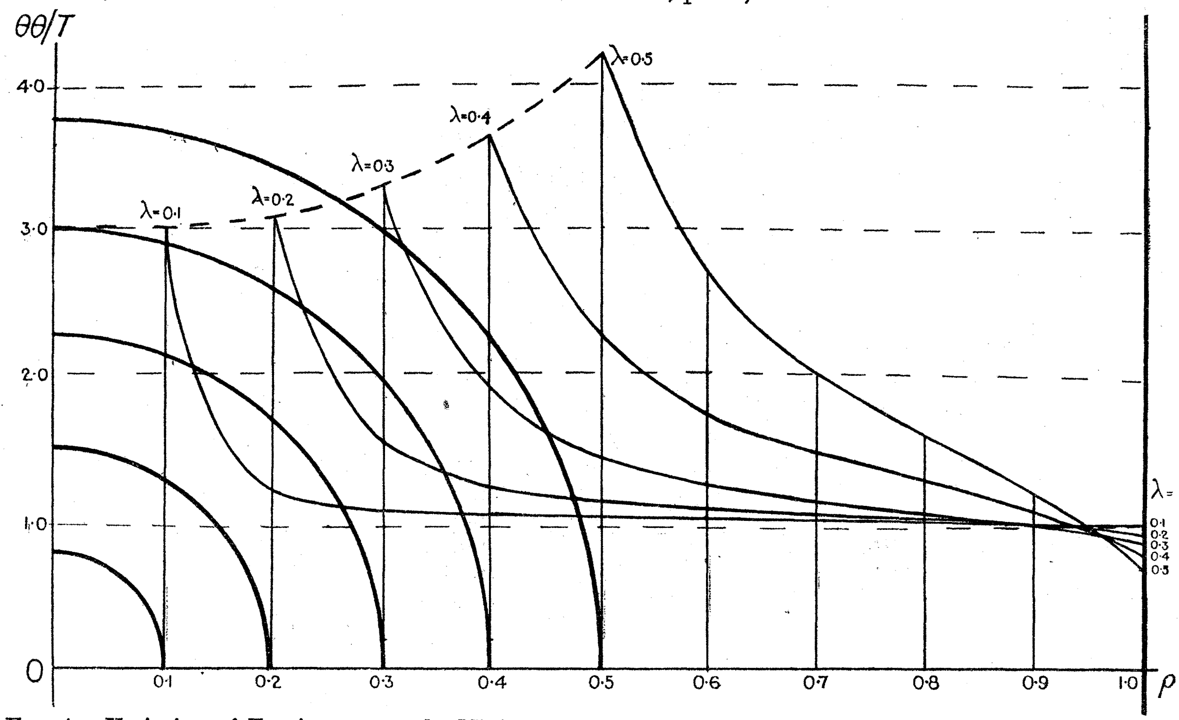

FIG. 4.—Variation of Tension across the Minimum Section for Different Values of λ. The Dotted Curve shows the Increase in the Maximum Stress Concentration as λ increases.

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Fig. 5 summarises all the results for a strip whose diameter is twice that of the hole ($\lambda = 0 \cdot 5$) and compares the calculated values with those obtained by experiment. The calculated values are taken from Tables XII to XVI, pp. 74, 75, 76, 78 and 81. The experimental results are taken from Table 1, p. 17, of the paper by Coker, Chakko and Satake cited above. The form of their results must be modified a little before a comparison can be made; their co-ordinates, x, y, must be doubled to correspond with the ξ, η of the present paper, and their absolute values of the stresses must be divided by the mean tension of 556 lbs./in.[2] In addition, they often measured the same stress by two distinct methods and also at two symmetrically placed points. Wherever this was done, a mean value has been taken.

The experimental results for the stresses round the hole have already been given in Table XIII. Those for the tensions across the minimum section and on the edge are recorded in Tables XVII and XVIII below. The calculated values from Tables XIV and XVI, pp. 76 and 81, are added for comparison.

The figures show clearly that the agreement between the calculated stresses and those found by experiment is usually close. Discrepancies, when they occur, are always in the neighbourhood of a boundary; but as the maximum and minimum stresses occur at the boundaries, these discrepancies are of importance. It has been suggested that they are to be explained by the difficulty of making optical measurements very close to an edge, a difficulty which will be increased if the stress varies rapidly from point to point. But, although this is the most likely explanation, it is not certainly the correct one. Another possibility is that the assumptions of the theory are not justified in the neighbourhood of the edge. Further experiments with a larger specimen and with special attention to the conditions near the boundaries should be able to settle this question.

Table XVII.—Stresses on the Minimum Section of a Strip ($\lambda = 0 \cdot 5$) by Experiment and by Calculation.

ξ	Experiment.		Calculation.
	$\widehat{xx}$.	$\widehat{xx}$/T.	$\widehat{xx}$/T.
	lbs./in.^{2}		
0·5	1,992	3·58	4·32
0·6	1,437	2·59	2·75
0·7	1,077	1·94	2·04
0·8	880	1·58	1·61
0·9	655	1·18	1·22
1·0	597	1·07	0·73

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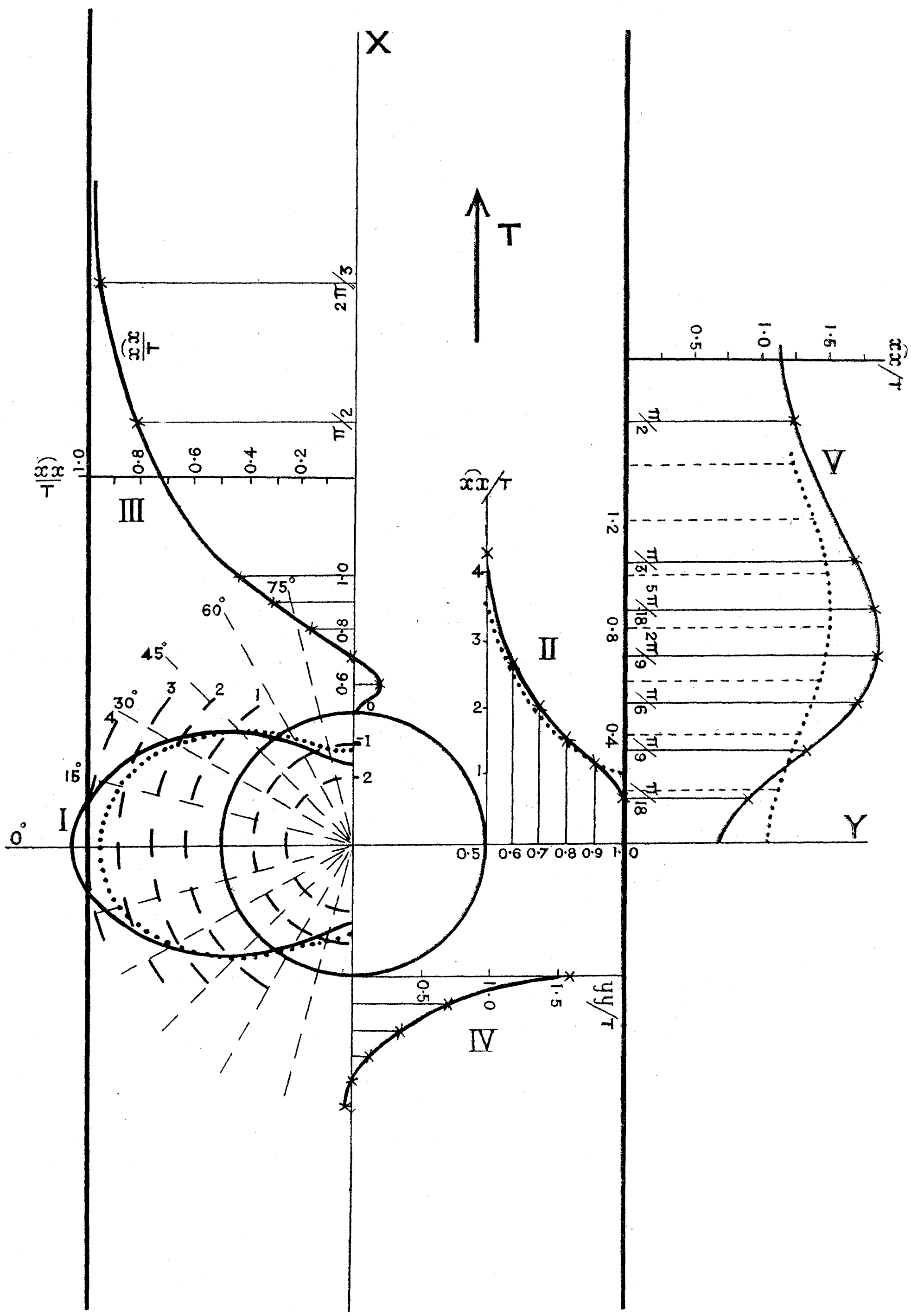

FIG. 5.—Stresses in a Strip containing a Hole of Diameter equal to half that of the Strip ($\lambda = 0{\cdot}5$). I, Stress at Rim of Hole, repeated from fig. 2; II, Stress on Minimum Section, repeated from fig. 4; III, Tension on the Axis (Table XV); IV, Cross-Stress on the Axis (Table XV); V, Tension on the Edge (Table XVI). The dotted curves give the values obtained by experiment.

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TABLE XVIII.—Tensions in the Edge of a Strip ($\lambda = 0 \cdot 5$) by Experiment and by Calculation.

ξ	Experiment.		Calculation.
	$\widehat{xx}$.	$\widehat{xx}$/T.	$\widehat{xx}$/T.
	lbs./in.^{2}		
0	595	1·07	0·71
$\pi/18 = 0 \cdot 17$	—	—	0·93
0·2	615	1·11	—
$\pi/9 = 0 \cdot 35$	—	—	1·38
0·4	690	1·24	—
$\pi/6 = 0 \cdot 52$	—	—	1·74
0·6	792	1·42	—
$2\pi/9 = 0 \cdot 70$	—	—	1·86
0·8	817	1·47	—
$5\pi/18 = 0 \cdot 87$	—	—	1·83
1·0	830	1·50*	—
$\pi/3 = 1 \cdot 05$	—	—	1·69
1·2	775	1·39	—
1·4	677	1·22	—
$\pi/2 = 1 \cdot 57$	—	—	1·24

* On p. 81 above the maximum value was stated as 1·6, and this is approximately correct for the greatest measured stress. The lower value in Table XVIII results from averaging in the way described.

In the preparation of the figures I have been greatly helped by the loan from Prof. E. G. COKER of a copy of his Paper, and by his permission to make use of his Tables; I record my thanks to him here.